CDS 101/110: Lecture 3.2
Linear Systems

Goals for Today:

- Review Convolution Integral and frequency response
- Summarize properties, examples, and tools
  - Convolution equation describing solution in response to an input
  - Step response, impulse response
  - Frequency response
- Characterize performance of linear systems

Reading:

- Åström and Murray, FBS-2e, Ch 6.1-6.3 (CDS 110: start 6.4)

The Convolution Integral

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

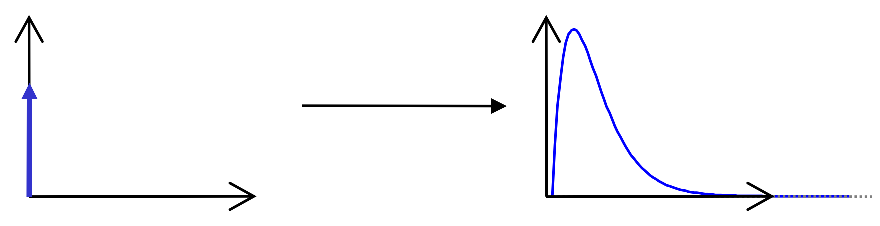
Summary of Derivation

- $x(t) = x_h(t) + x_p(t)$
- $x_h(t) = e^{At}x(0)$
- Find $x_p(t)$ for case of impulse input: $u(t) = \delta(t)$
  - $x_p(0^+) = \int_{0^-}^{0^+} \dot{x}dt = \int_{0^-}^{0^+} (Ax(t) + B\delta(t))dt = B \quad \rightarrow$

    $$x_p(t) = e^{At}B; \quad y(t) = Ce^{At}B$$

  - ***Aside***: $h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(\tau) = Ce^{At}B + D\delta(\tau)$

    is the *impulse function* of (A,B,C,D)



The Convolution Integral

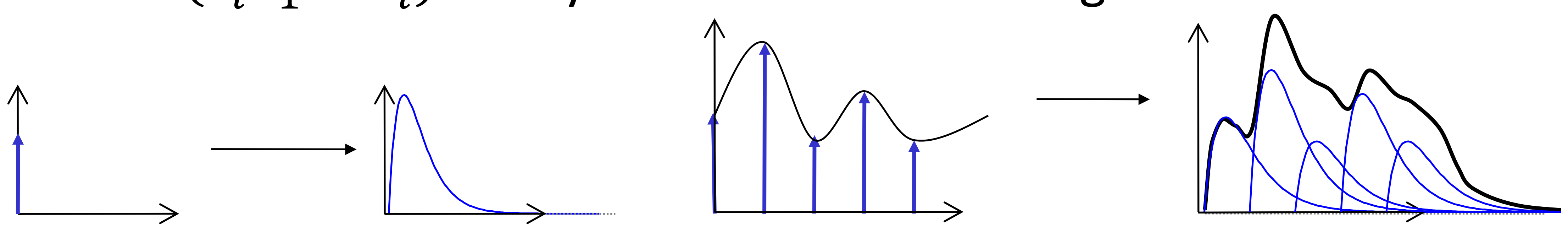
$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Summary of Derivation

– Find $x_p(t)$ for impulse series: $u(t) = c_1\delta(t-\tau_1) + c_2\delta(t-\tau_2) + \cdots$

$$x_p(t) = c_1 e^{A(t-\tau_1)}B + c_2 e^{A(t-\tau_2)}B + \cdots$$

– Limit as $(\tau_{i-1} - \tau_i) \rightarrow 0$ yields convolution integral



– Note FBS "proves" that convolution integral is solution to

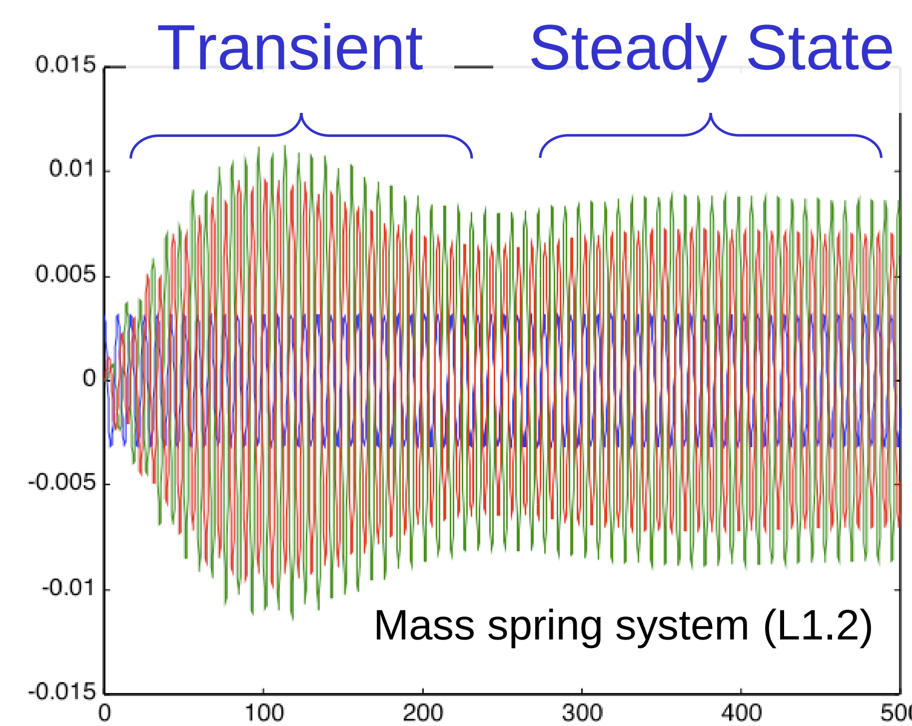
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

By substitution.

– See A. Lewis, *A Mathematical Introduction to Classical Control, p. 48+*

Input/Output Performance

- How does system respond to changes in input values?
  - Transient response:
  - Steady state response:
- Characterize response in terms of
  - Impulse response
  - Step response
  - Frequency response



$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Response to I.C.s, Transient response

Response to inputs, Steady State (if constant inputs)

Step Response

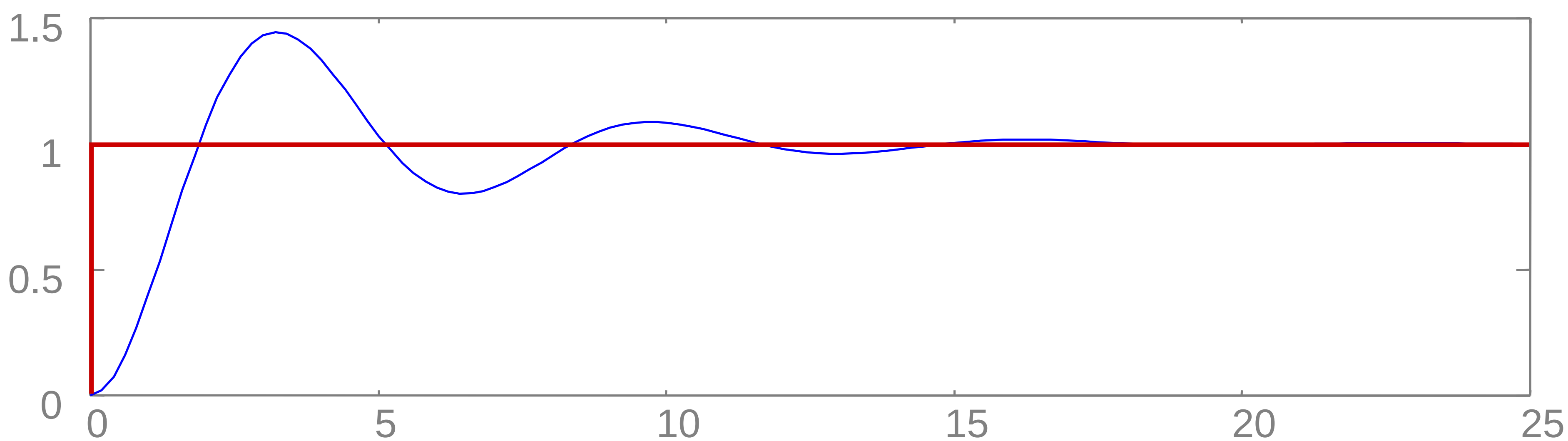
- Output characteristics in response to a "step" input
  - If $x(0) = 0$, and *unit* step input

$$u(t) = H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

  - Then convolution integral yields:

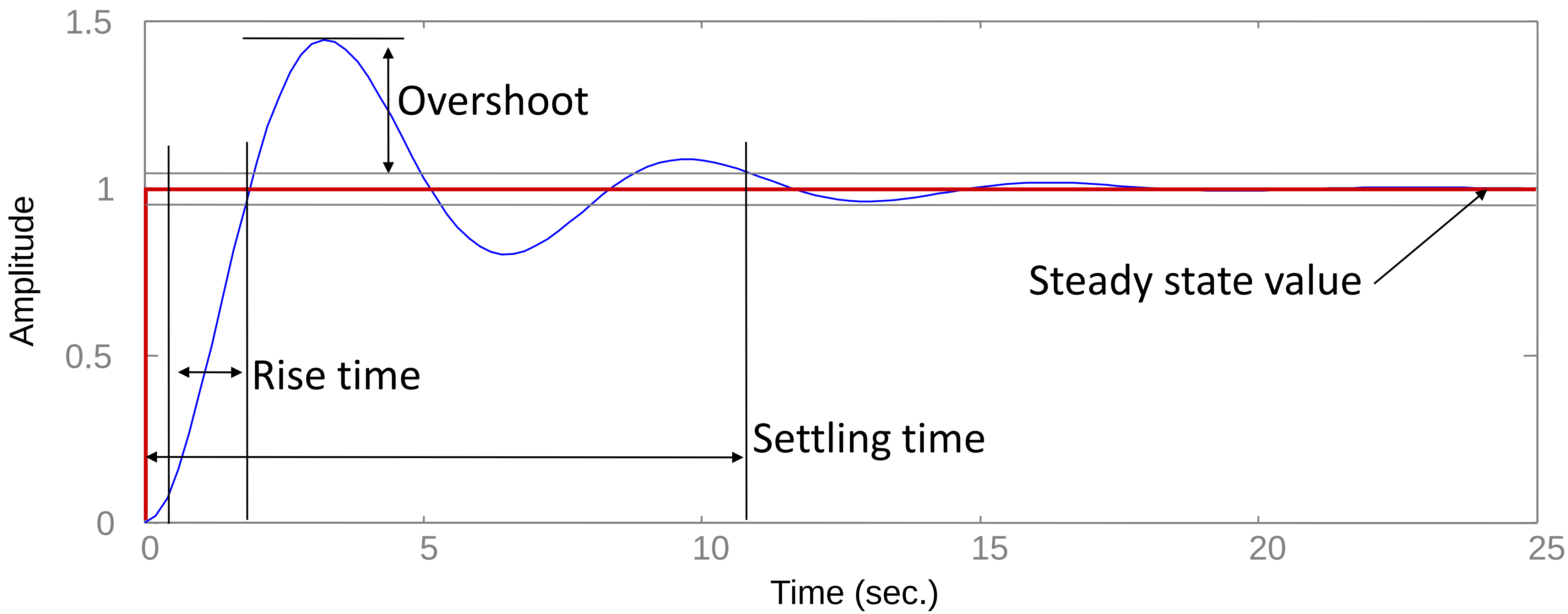
$$y(t) = \int_0^t Ce^{A(t-\tau)}\,Bd\tau + D = C\int_0^t e^{A\sigma}\,Bd\sigma + D = C\left(A^{-1}e^{A\sigma}B\right)\Big|_{\sigma=0}^{\sigma=t}$$

$$= \underbrace{CA^{-1}e^{At}B}_{\text{Transient}} - \underbrace{CA^{-1}B + D}_{\text{Steady State}}$$



Step Response

- Output characteristics in response to a "step" input
  - **Rise time:** time required to move from 5% to 95% of final value
  - **Overshoot:** ratio between amplitude of first peak and steady state value
  - **Settling time:** time required to remain w/in *p*% (usually 2%) of final value
  - **Steady state** value: final value at $t \to \infty$



Calculating Frequency Response from convolution equation

- Convolution equation describes LTI system response to any input; e.g., response to sinusoidal input: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$u(t)$

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B \boxed{e^{i\omega\tau}} \mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}\int_0^t e^{(i\omega I - A)\tau} B \mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1} e^{(i\omega I - A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}\left(e^{(i\omega I - A)t} - I\right) B \\
&= \underbrace{e^{At}\left(x(0) - (i\omega I - A)^{-1}B\right)}_{\text{Transient (decays if stable)}} + \underbrace{(i\omega I - A)^{-1}B}_{\text{Ratio of response/input}} e^{i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + \boxed{\left(C(i\omega I - A)^{-1}B + D\right)} e^{i\omega t}
\end{aligned}
$$

"Frequency response"

Calculating Frequency Response from convolution equation #2 (more in 2 weeks)

- More generally, consider: $u(t) = e^{\sigma t} e^{i\omega t} = e^{(\sigma + i\omega)t} = e^{st}$

u(t)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau \\
&= e^{At}x(0) + e^{At} \int_0^t e^{(sI-A)\tau} B d\tau \\
&= e^{At}x(0) + e^{At}(sI-A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right) B \\
&= \underbrace{e^{At}\left(x(0) - (sI-A)^{-1}B\right)} + \underbrace{(sI-A)^{-1}Be^{st}}
\end{aligned}
$$

"Transient" (decays…) Ratio of response/input

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (sI-A)^{-1}B\right) + \left(C(sI-A)^{-1}B + D\right)e^{st}
\end{aligned}
$$

"Transfer Function"

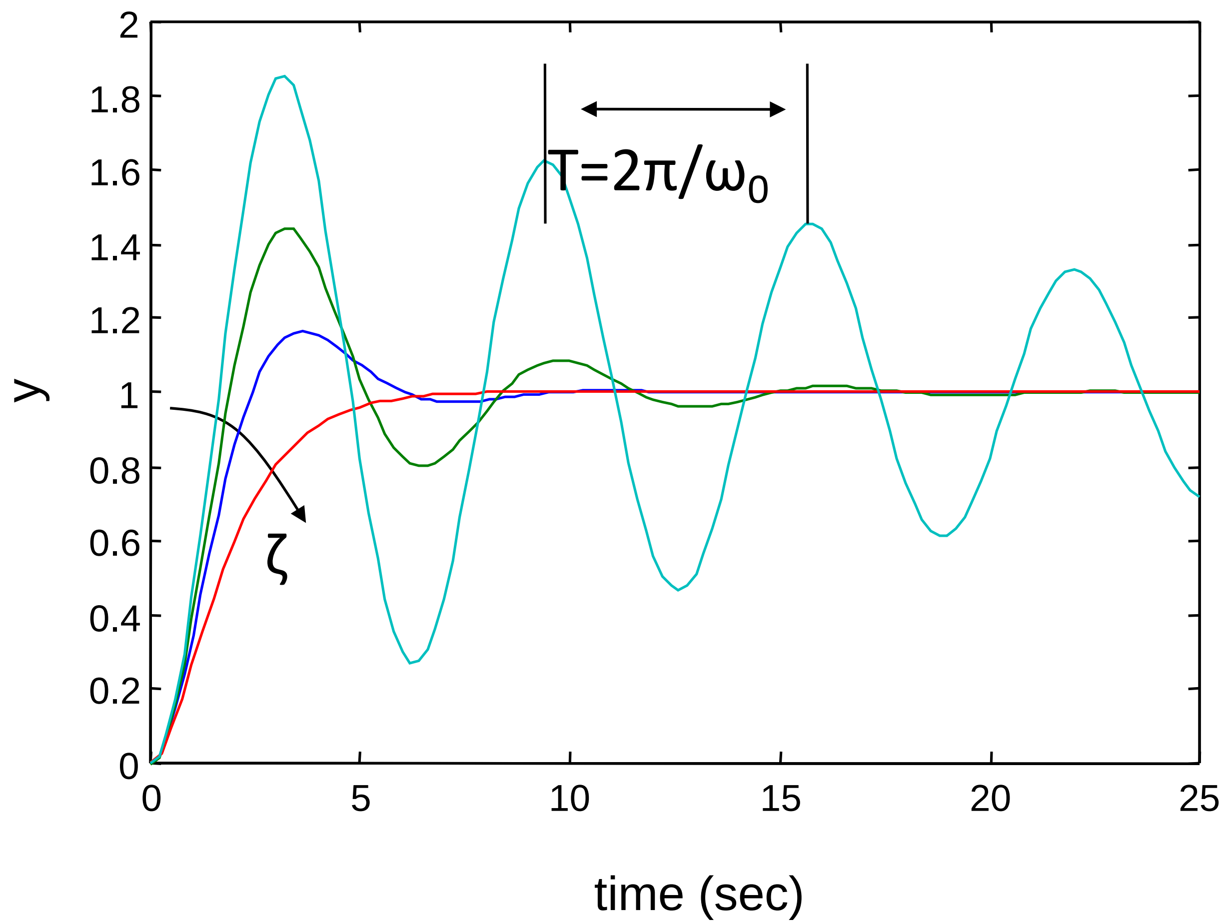
$y_{ss}(t) = Me^{i\theta}e^{st}$ where $Me^{i\theta} = C(sI-A)^{-1}B + D$

- M represents magnitude (or *gain)* of $C(sI-A)^{-1}B + D$
- θ represents phase, since if $u(t) = cos\omega t$, $y_{ss}(t) = Mcos(\omega t + \theta)$

Second Order Systems

Many important examples, and Insight to higher order systems

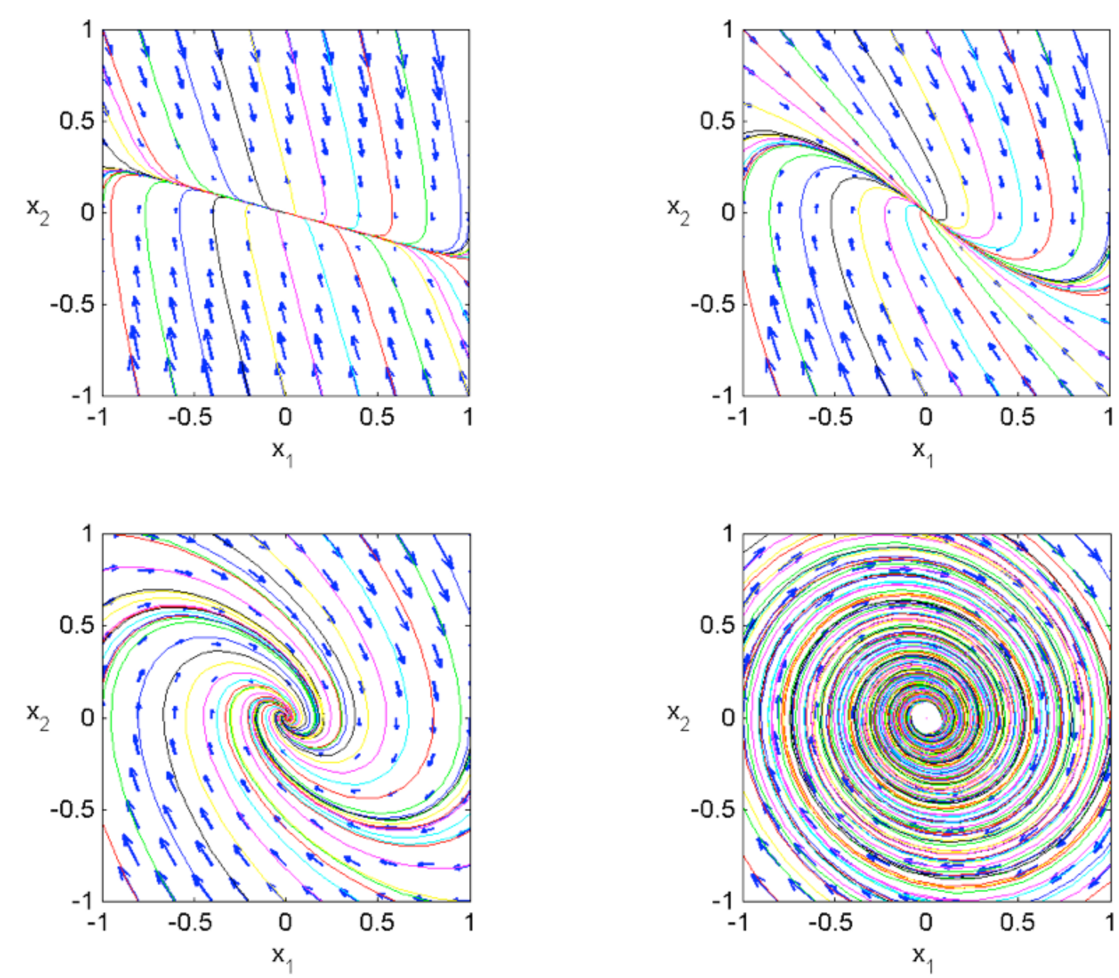
$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \qquad \leftrightarrow \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



- Analytical formulas exist for overshoot, rise time, settling time, etc
- Will study more next week

For $\zeta < 1$, eigenvalues at

$$\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$



Second Order Systems

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

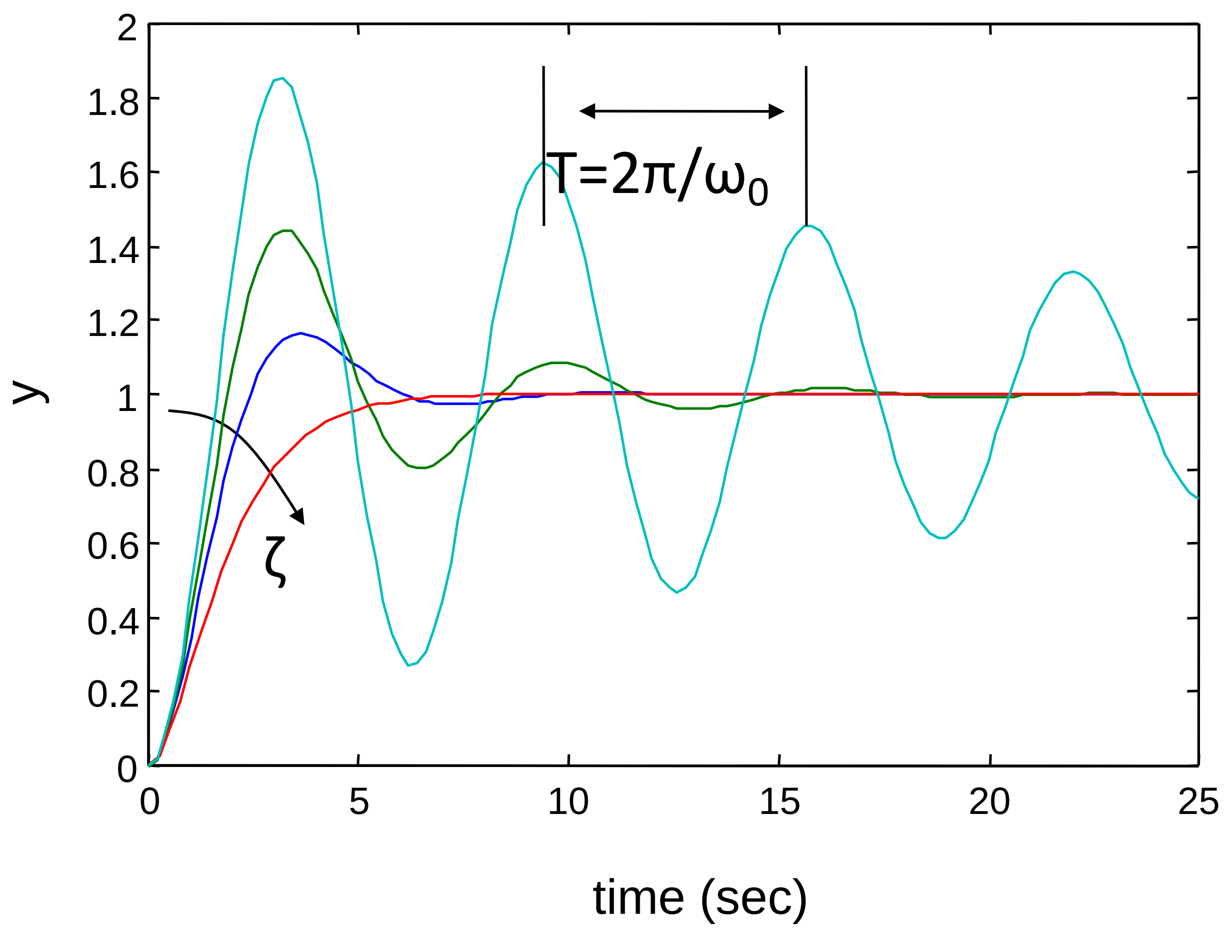
- Impulse response: $C = [1\ 0]$

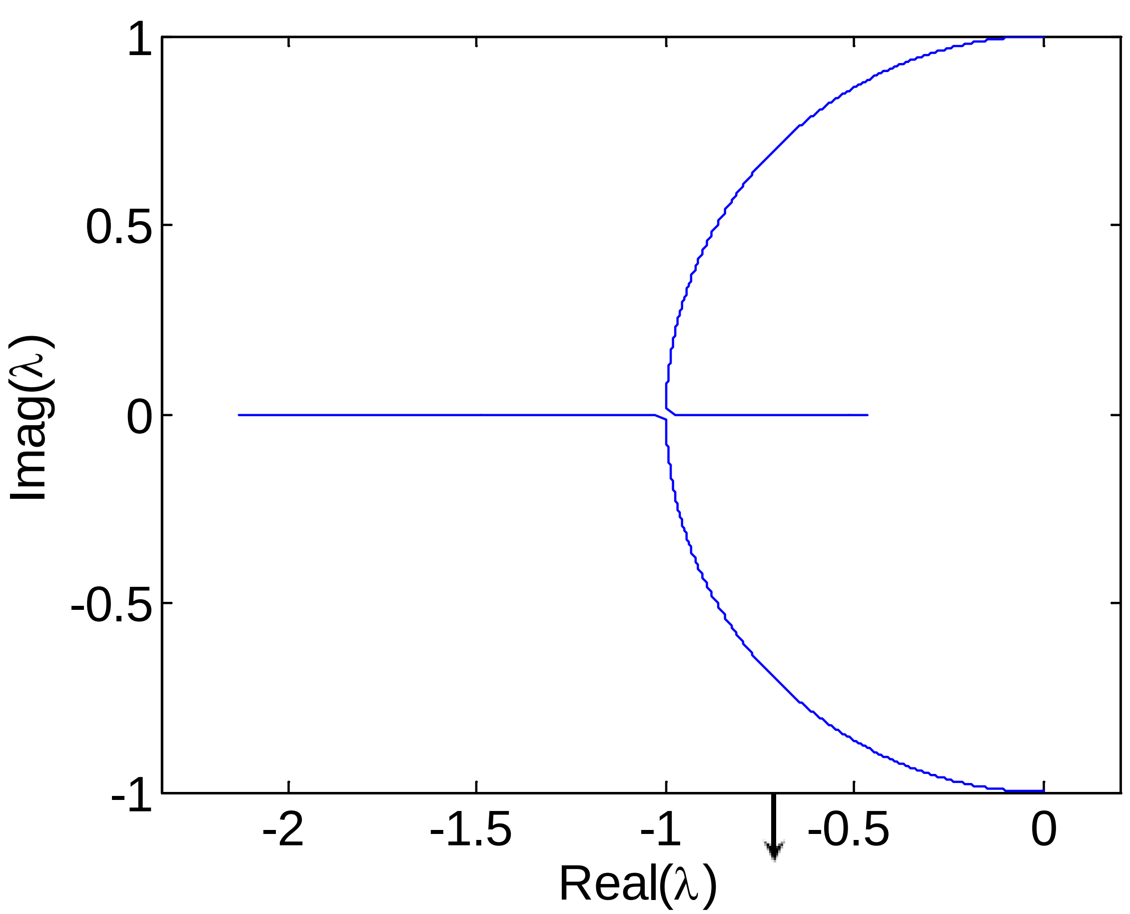
$$h(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin \omega_d t$$

For $\zeta < 1$, eigenvalues at

$$s_{1,2} = \left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$

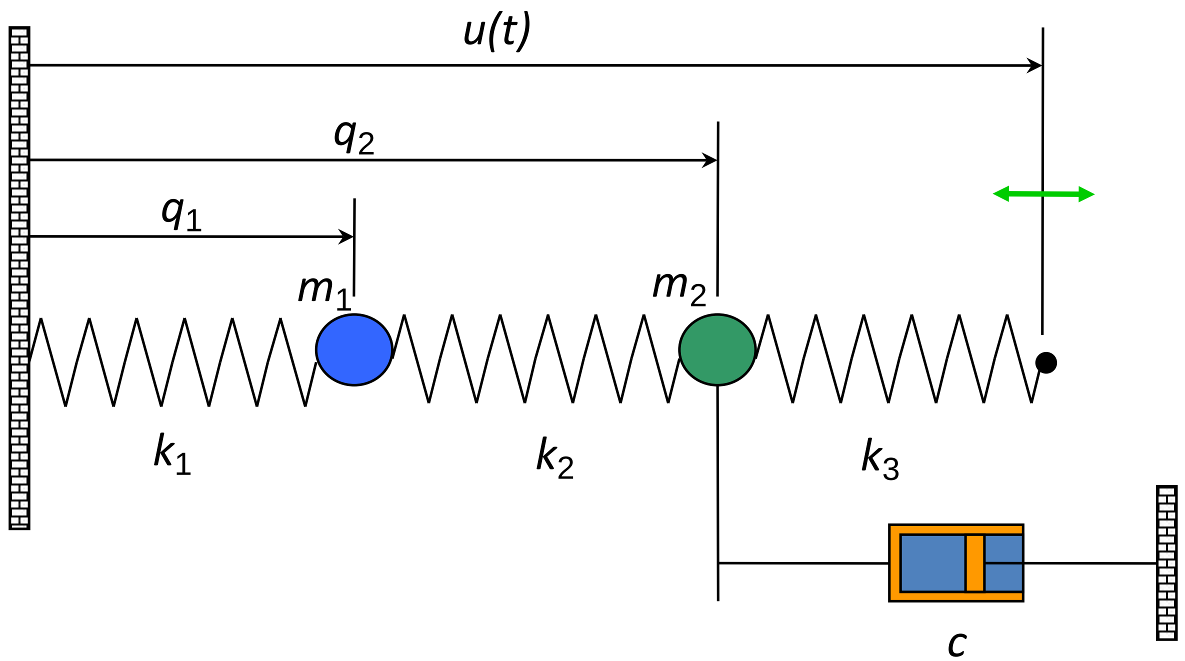
- Step response:





Spring Mass System

Frequency response: $C(j\omega I-A)^{-1}B+D$



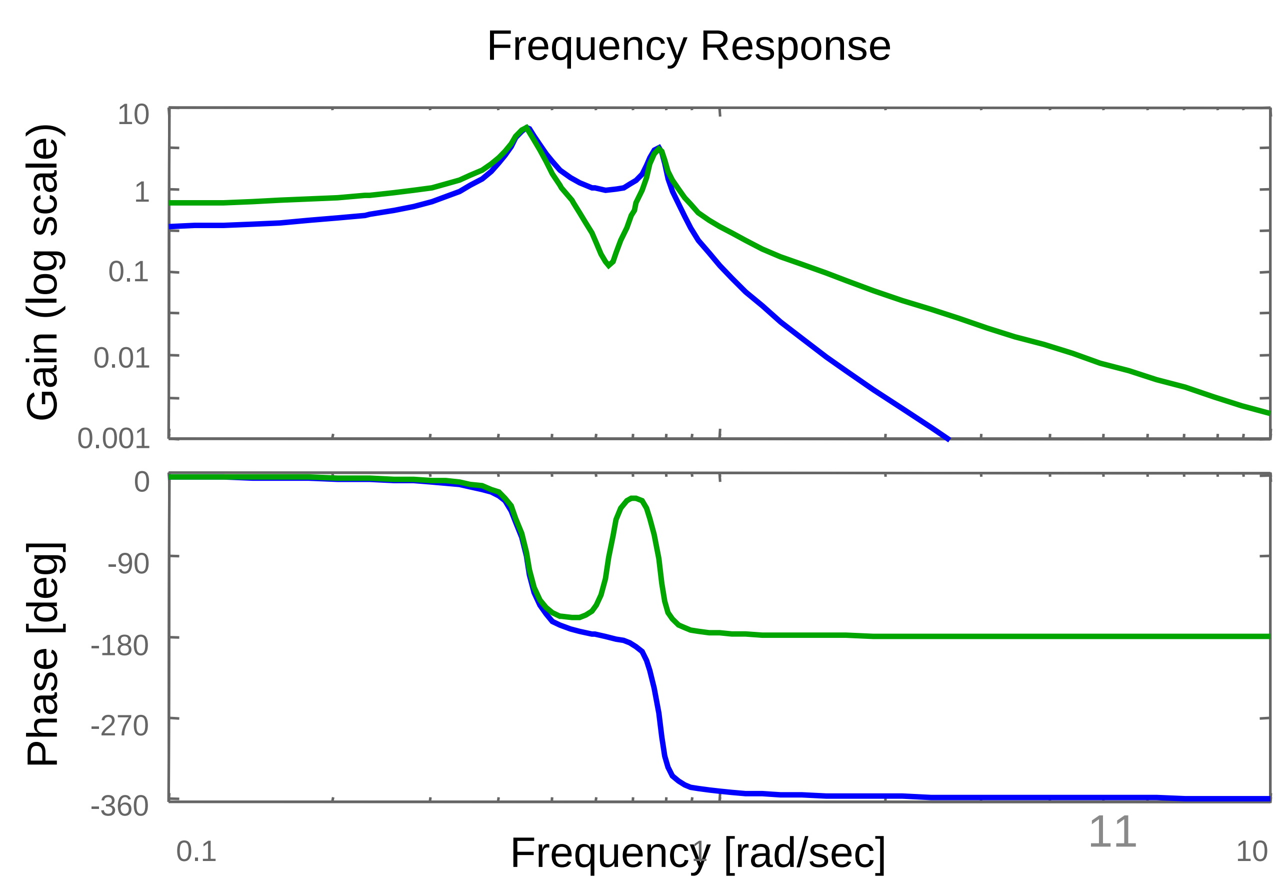
$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

With $k_1 = k_2 = 1,\ m = 1,\ c = 0$

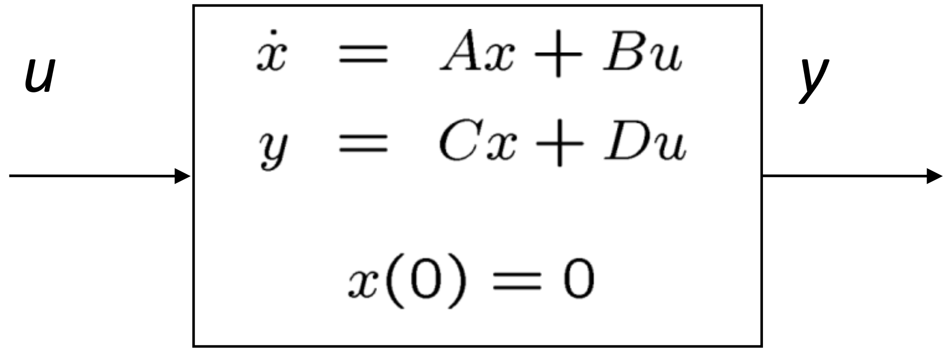
$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \qquad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

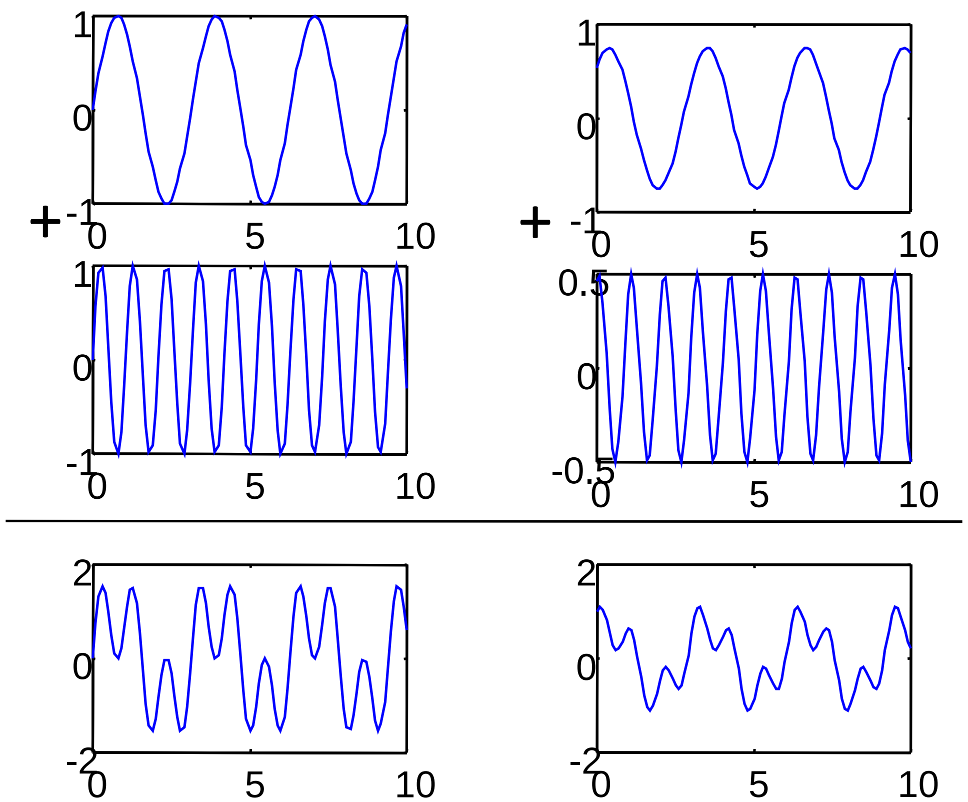
Eigenvalues of A:

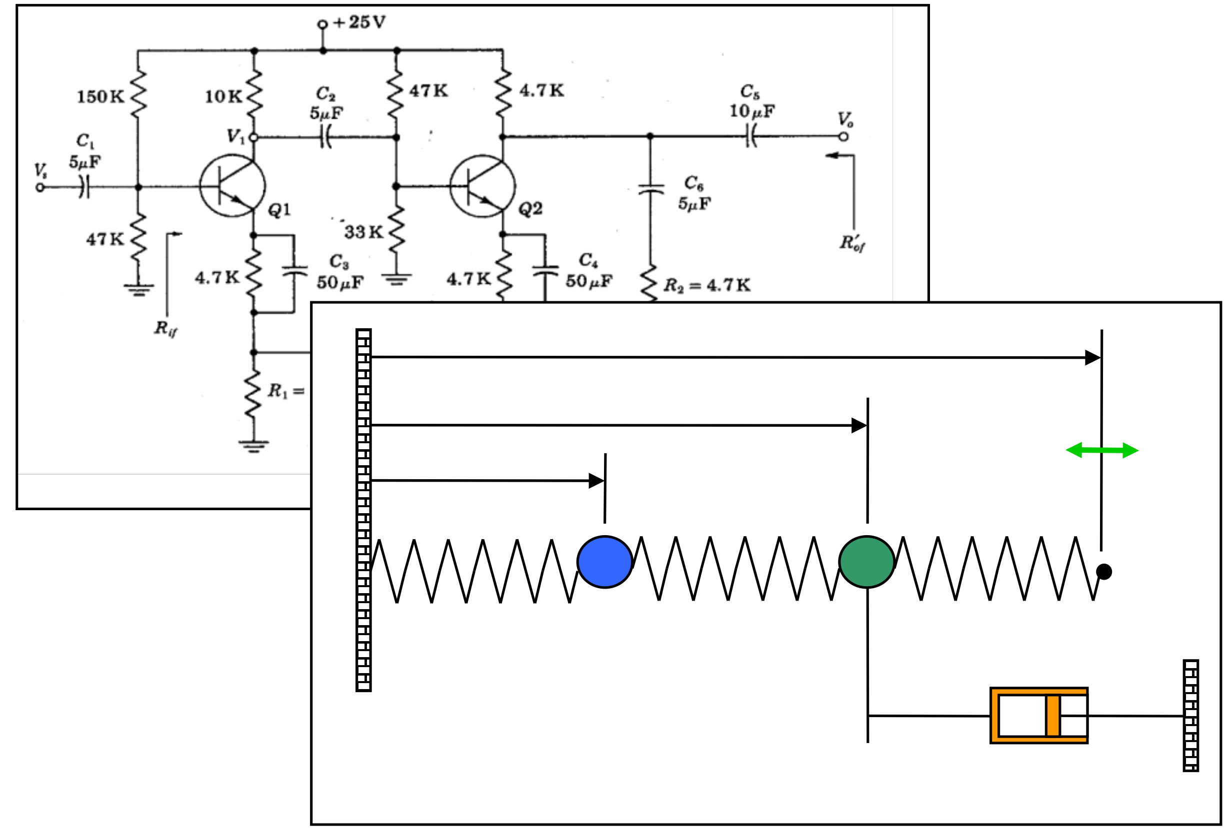
- For zero damping, $j\omega_1$ and $j\omega_2$
- ω_1 and ω_2 correspond frequency response peaks
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for lower freq.
  - Out-of phase motion for higher freq.



Summary: Linear Systems



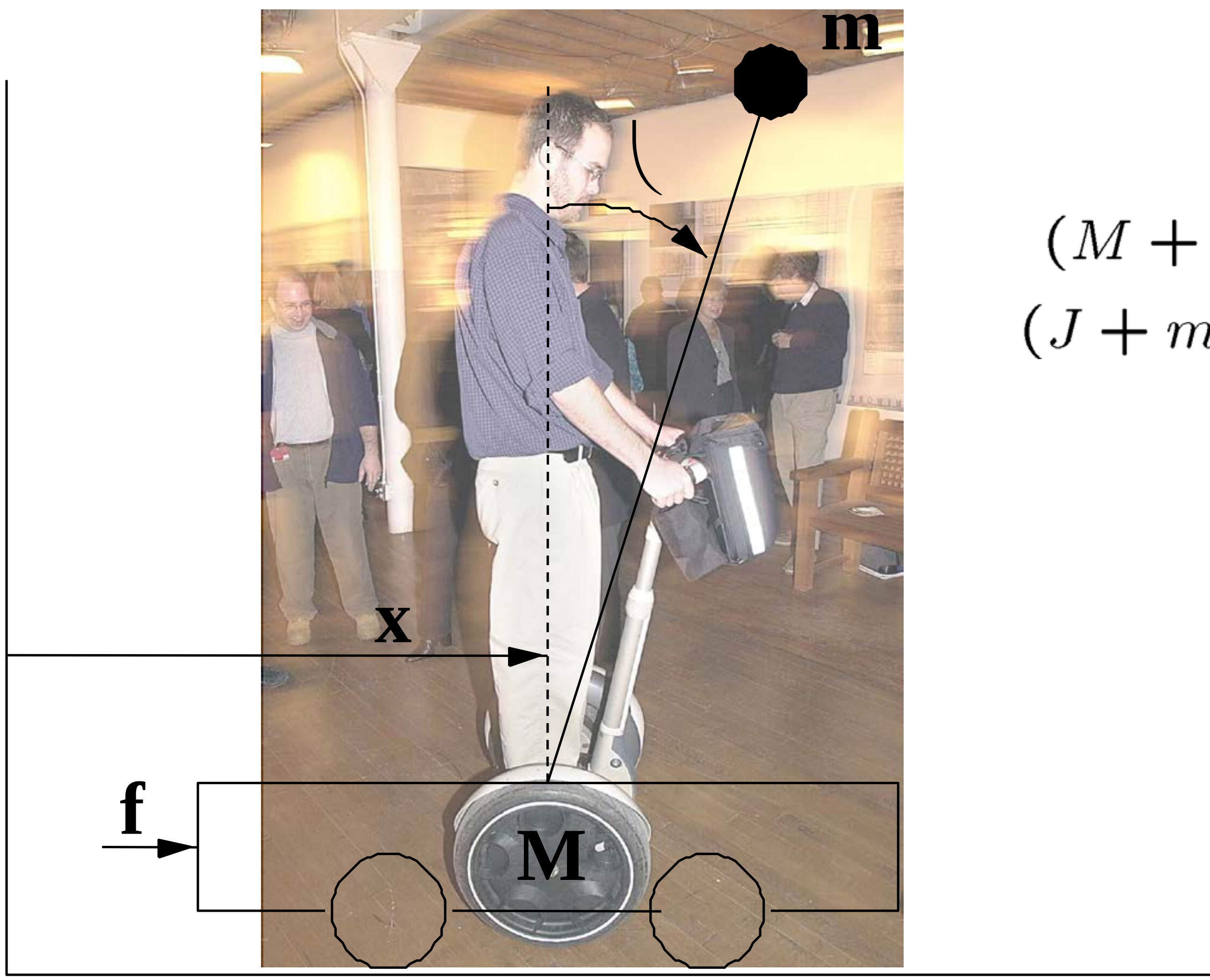




$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- Properties of linear systems
  - Linearity with respect to initial condition and inputs
  - Stability characterized by eigenvalues
  - Convolution Integral describes output
  - Many applications and tools available
  - Provide local description for nonlinear systems

Example: Inverted Pendulum on a Cart



$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around θ = 0

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{J(M+m)+Mml^2} & \dfrac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \dfrac{mgl(M+m)}{J(M+m)+Mml^2} & \dfrac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{J(M+m)+Mml^2} \\ \dfrac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$