ME/CS 133(a): Solution to Homework #1
(Fall 2017/2018)

Solution to Problem 1:

Let the 2×1 vectors ${}^1\vec{v} = \begin{bmatrix} {}^1v_x & {}^1v_y \end{bmatrix}^T$ and ${}^2\vec{v} = \begin{bmatrix} {}^2v_x & {}^2v_y \end{bmatrix}^T$ have associated complex representations ${}^1\tilde{v} = {}^1v_x + i\ {}^1v_y$ and ${}^2\tilde{v} = {}^2v_x + i\ {}^2v_y$ respectively (where $i^2 = -1$). Recall that the goal of this problem is to show that the complex number formula:

$$ {}^1\tilde{v} = \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v}\ . \tag{1} $$

is equivalent to the planar coordinate transformation:

$$ {}^1\vec{v} = \vec{d}_{12} + R(\theta_{12})\ {}^2\vec{v}\ . \tag{2} $$

Let's evaluate the right hand side of expression (1) using the standard rules for multiplication of complex numbers[1]:

$$ \begin{aligned} \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v} &= (x + iy) + (\cos\theta_{12} + i\sin\theta_{12})({}^2v_x\ +\ i\ {}^2v_y) \\ &= (x +\ {}^2v_x\cos\theta_{12}\ -\ {}^2v_y\sin\theta_{12}) + i(y +\ {}^2v_x\sin\theta_{12}\ +\ {}^2v_y\cos\theta_{12}) \end{aligned} \tag{3} $$

where we have used Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$). Matching the real and complex portions of Equation (3) with the real and complex parts of ${}^1\tilde{v}$ in the left hand side of Equation (1), we see that

$$ {}^1v_x = x + {}^2v_x\cos\theta - {}^2v_y\sin\theta \tag{4} $$
$$ {}^1v_y = y + {}^2v_x\sin\theta + {}^2v_y\cos\theta\ . \tag{5} $$

These equations are equivalent to

$$ {}^1\vec{v} = \vec{d}_{12} + \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{bmatrix}\ {}^2\vec{v} \tag{6} $$

Solution to Problem 2: Recall that the location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position ${}^B\vec{p}$ as seen by an observer in the fixed B frame. After displacement, the observer in the body fixed C frame also sees the pole in his/her coordinates at point ${}^B\vec{p}$. However, the moving body has displaced relative to the fixed observer by amount $D_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by a coodinate transform. Since the pole is at the same location in both the fixed and moving frames, it must be true that:

$$ {}^B\vec{p} = \vec{d}_{12} + R_{12}\ \ {}^B\vec{p}. $$

This equation can be solved to find the pole location:

$$ {}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} $$

[1] If $\tilde{a} = a_1 + ia_2$ and $\tilde{b} = b_1 + ib_2$, then $\tilde{a}\tilde{b} = (a_1b_2 - a_2b_2) + i(a_1b_2 + a_2b_1)$.

Of course, the matrix $(I - R_{12})$ must be invertible,which will alwas be true except when $R_{12} = I$. In this case, the motion is a pure translation, which is viewed as a rotation about the "pole at infinity."

B) In Frame B, the pole is located at: ${}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12}$

C) In Frame C, the vector describing the pole has exactly the same value as seen by the observer in Frame B: ${}^C\vec{p} = (I - R_{12})^{-1}\vec{d}_{12}$

A) In Frame A, the expression for the pole vector is obtained by a simple coordinate transformation of the expression in Frame B: ${}^A\vec{p} = \vec{d}_{01} + R_{01}\ {}^B\vec{p} = \vec{d}_{01} + R_{01}(I - R_{12})^{-1}\vec{d}_{12}$

Problem 3: To find the pole of the displacement: $D_2 = (x, y, \theta) = (2.0, 2.0, 45.0^o)$, substitute into the above results:

$$\begin{aligned}
{}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} &= \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\cos(45^o) & -\sin(45^o)\\ \sin(45^o) & \cos(45^o)\end{pmatrix}\right]^{-1}\begin{bmatrix}2.0\\ 2.0\end{bmatrix} \\
&= \begin{bmatrix}1-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2}\end{bmatrix}^{-1}\begin{bmatrix}2.0\\ 2.0\end{bmatrix} \\
&= \begin{bmatrix}-1.41421\\ 3.4142\end{bmatrix} \qquad (7)
\end{aligned}$$

You could report this result in Frame B, or transform the results to frame A.

$$\begin{aligned}
{}^A\vec{p} = \vec{d}_{01} + R_{01}\ {}^B\vec{p} &= \begin{bmatrix}1.0\\ 2.0\end{bmatrix} + \begin{pmatrix}\cos(30^o) & -\sin(30^o)\\ \sin(30^o) & \cos(30^o)\end{pmatrix}\begin{bmatrix}-1.414215\\ 3.4142\end{bmatrix} \qquad (8) \\
&= \begin{bmatrix}-1.9319\\ 4.2497\end{bmatrix} \qquad (9)
\end{aligned}$$

Problem 4: To show that a transformation is a pure rotation when viewed in a reference frame at the pole, select a new reference frame, denoted by D, whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement. Let $\vec{p}$ denote the location of the pole, as seen by an observer in Frame B. The location of Frame B relative to Frame D is a pure translation of amount $\vec{p}$, and therefore, $D_{DB} = (-\vec{p}, I)$. The displacement of the body from the first position to the second position, as now observed in Frame D, is obtained by a similarity transform $D_{DB}D_{12}D_{DB}^{-1}$:

$$\begin{aligned}
D_{DB}D_{12}D_{DB}^{-1} &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(-\vec{p}, I)^{-1} \qquad (10) \\
&= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(+\vec{p}, I) \qquad (11) \\
&= (-\vec{p}, I)((\vec{d}_{12} + R_{12}\vec{p}), R_{12}) \qquad (12) \\
&= ((\vec{d}_{12} + (R_{12} - I)\vec{p}), R_{12}) \qquad (13)
\end{aligned}$$

Hence, if $\vec{p} = -(R_{12} - I)^{-1}\vec{d}_{12} = (I - R_{12})^{-1}\vec{d}_{12}$, then $D_{DB}D_{12}D_{DB}^{-1} = (\vec{0}, R_{12})$. I.e., as viewed in reference Frame D, the displacement is a pure rotation by amount R_{12}.

Problem 5:

Part (a): There are many ways that one can prove that reflections preserve length. Here is one approach (see Figure 1).

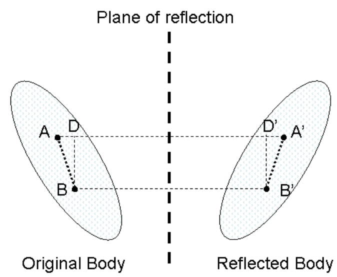

Figure 1: Geometry of Planar Rigid Body Reflection

Select any two non-identical points, A and B, in a rigid body. After reflection, those points become $A^{'}$ and $B^{'}$. Form the right triangle ABD, where the line BD is chosen to be perpendicular to the line $AA^{'}$. Similarly, in the reflected body, form the right triangle $A^{'}B^{'}D^{'}$. Simple geometric arguments show that since the distance $|BD|$ and $|B^{'}D^{'}|$ are equal, and the distances $|AD|$ and $|A^{'}D^{'}|$ are equal, then $|AB| = |A^{'}B^{'}|$. Hence, the distance between A and B is preserved under reflection. Since A and B were chosen randomly, the result will hold for any non-identical pair of points in the body. Thus, distance is always preserved under reflection.

Part (b): Generally, physically meaningful planar displacements are not equivalent to a single reflection. To see this, define three points (A, B, C) in the body of Figure 1. Because the body is rigid, one can think of points (A, B, C) as forming a rigid triangle. Consider the triangle formed from the reflected points $(A^{'}, B^{'}, C^{'})$. Note that it is impossible physically translate (A, B, C) to $(A^{'}, B^{'}, C^{'})$. Finally, note that any rigid body planar displacement can generally be realized as the result of two sequential reflections.

An alternative proof for problem 5:

Part (a):

Without loss of generality, we can select any coordinate system on the plane. We choose an xy-coordinate system such that the y-axis is coincident with the line of reflection. Under this coordinate system, the reflection of any point (x, y) has coordinates $(-x, y)$. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two points on the rigid body, with corresponding reflections A' and B' respectively. Then, $A' = (-x_1, y_1)$ and $B' = (-x_2, y_2)$. To see that $|AB| = |A'B'|$, we can just plug their coordinates into the distance formula: $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |A'B'| = \sqrt{(-x_2 - (-x_1))^2 + (y_2 - y_1)^2}$.

Part (b):

As in part (a), without loss of generality, we can select an xy-coordinate system on the plane such that the y-axis is coincident with the line of reflection. Under this coordinate system, the reflection of any point (x, y) has coordinates $(-x, y)$. Let D be the reflection operator, so that $D : (x, y) \to (-x, y)$.

We prove by contradiction that operator D cannot be a planar displacement operator. Assume that D is represented by some planar displacement operator, such that $D = (\mathbf{d}, R)$ for some vector $\mathbf{d}$ and rotation matrix R.

Let B be the set of points on the body. We assume that there exists a point $P_0 \in B$ and a number $\epsilon > 0$ such that $N_\epsilon(P_0) = \{X \in \mathbb{R}^2 | \; ||X - P_0||_2 < \epsilon\} \subseteq B$. This just means that there exists some open set that is contained inside B. Letting $P_0 = (x_0, y_0)$, there must exist some points $P_1 = (x_1, y_0), P_2 = (x_0, y_1)$, and $P_3 = (x_1, y_1)$, where $x_0 \neq x_1$ and $y_0 \neq y_1$. For example, if you set $x_1 = x_0 + \frac{\epsilon}{4}$ and $y_1 = y_0 + \frac{\epsilon}{4}$, then clearly $\{P_0, P_1, P_2, P_3\} \subset N_\epsilon(P_0) \subseteq B$.

The reflections of these points under our reflection operator are $P_0' = (-x_0, y_0), P_1' = (-x_1, y_0), P_2' = (-x_0, y_1), P_3' = (-x_1, y_1)$.

Next, we note that D cannot represent a pure translation, i.e. $R \neq I$. This is because under a pure translation, each point must have an equal distance of displacement under the operator; however, $|P_0 P_0'| = 2|x_0| \neq |P_1 P_1'| = 2|x_1|$. This means that D must have a finite pole; in other words, the pole of the planar displacement is *not* located at infinity.

Consider the displacements $P_0 \to P_0'$ and $P_1 \to P_1'$. Knowing the movement of these points under D is enough to fully define the operator D: as discussed in class, a planar displacement has 3 degrees of freedom. The movement of these 2 points is a pure rotation around the point $\overline{P} = (0, y_0)$. We can see that this must be the pole of the displacement, as (1) the pole is unique for any displacement that is not a pure translation, and (2) the reflections P_0' and P_1' are both achieved by rotating P_0 and P_1 about $\overline{P} = (0, y_0)$ by an angle of π radians.

Consider the displacements $P_2 \to P_2'$ and $P_3 \to P_3'$. By the same logic as that of the previous paragraph, the pole of the displacement must be at $(0, y_1)$ in order for the movement of these 2 points to be a pure rotation about the pole.

Thus, we have found that the pole of the displacement operator is at $\overline{P} = (0, y_0) = (0, y_1)$. Since $y_0 \neq y_1$, we have arrived at a contradiction! Thus, no planar displacement can equivalently perform a reflection.