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Linear observer synthesis for nonlinear systems using Koopman Operator framework

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Abstract: In this paper we develop a new approach for observer synthesis for discrete time autonomous nonlinear systems based on Koopman operator theoretic framework. Koopman operator is a linear but an infinite-dimensional operator that governs the time evolution of system outputs in a linear fashion. We exploit this property to synthesize an observer form which enables the use of Luenberger/Kalman-like linear observers for nonlinear estimation. Using the techniques for Koopman eigenvalue/eigenfunction/mode computation, we describe a numerical procedure to construct such an observer form which is often valid in a large portion of state space or even globally. We numerically compare our approach with Extended Kalman Filter and report superior performance.

Keywords: Nonlinear Dynamics, Operator Theory, Observer Design

1. INTRODUCTION

In this paper we develop a new approach for observer synthesis for nonlinear systems based on Koopman operator theoretic framework. Observer design for nonlinear systems is an extensively researched area, see Misawa and Hedrick (1989); Nijmeijer and Fossen (1999); Besancon (2007). Extended Kalman Filter (EKF) is most widely used nonlinear observer, its virtue being relative simplicity and frequently good performance. EKF being based on linearization is unfortunately not guaranteed to converge. In this paper we will focus on observer design (via e.g.global/pseudo/extended linearization, linearization by input-output injection, immersion etc.) based on *observer forms* (Misawa and Hedrick (1989); Keller (1987); Respondek (2001); Kang et al. (2013)), which not only enables exploiting Luenberger/Kalman-like linear observers in context of nonlinear estimation, but also often enjoy reliable performance guarantees in a large portion of state space or even globally. However, these techniques are often very restrictive, and the transformation which converts the system into a suitable observer form is very difficult to compute, making these approaches accessible only in a very narrow context. In this paper we exploit the spectral properties of Koopman operator to construct a similar transformation in a more general setting, thus potentially making linear observer like design accessible in a much broader context for nonlinear estimation.

Koopman operator is a linear but an infinite-dimensional operator that governs the time evolution of observables or outputs defined on the state space of a dynamical system, see Mezic (2012); Budisic et al. (2012). Koopman operator being linear admits eigenvalues and eigenfunctions, and enables one to express time evolution of the system outputs as a linear superposition of Koopman modes. Recent advances in techniques for computing Koopman spectral properties have fueled its application in several domains such as fluid mechanics (Rowley et al. (2009); Chen et al. (2012); Mezic (2012)), building data visualization (Eisenhower et al. (2010)), power system stability analysis (Susuki and Mezic (2014)), data fusion (Williams et al. (2015b)), and computer vision (Grosek and Kutz (2013); Surana (2015)), to name a few. These applications largely exploit Koopman framework for data driven model reduction along similar lines such as Proper Orthogonal Decomposition (Rowley et al. (2009)).

There has also been some exploration in using operator theoretic techniques for other system and controls applications such as stability analysis. For instance, Perron Frobeniuos operator (which is adjoint of Koopman operator) based approaches have been proposed for nonlinear stability analysis (Vaidya and Mehta (2008); Vaidya (2015)), computing domain of attraction (Wang and Vaidya (2010)), defining/constructing observability gramian for nonlinear systems (Vaidya (2007b)) and for sensor/actuator placement (Vaidya et al. (2011)). Key concept here is notion of Lyapunov measure equation (analogous to matrix Lyapunov equation, see Vaidya (2007a)) which provides necessary and sufficient condition for almost every where stability of an invariant set in nonlinear systems. Along similar lines, necessary and sufficient relationships between the existence of specific eigenfunctions of the Koopman operator and the global stability property of fixed points and limit cycles have been established in (Mauroy and Mezic (2013, Unpublished)). Numerical methods to estimate the region of attraction of the fixed point/limit cycle were also discussed. The use of infinite dimensional constructions/representations as discussed above have also been used in other contexts in systems and control applications, see Kreisselmeier and Engel (2003); Lasserre et al. (2008); Henrion and Korda (2014).

In this paper, we introduce a new notion of Koopman Observer Form (KOF) for a nonlinear discrete time autonomous systems with outputs. KOF is linear time invariant system with outputs, and is constructed from Koopman eigenvalues/eigenfunctions for the underlying nonlinear system, and Koopman modes for the system state and the outputs. It is important to note that while Koopman operator has infinitely many eigenfunctions (and eigenvalues), but for construction of KOF only a specific subset, whose span contain the system state and outputs, is required. KOF being linear, enables the use of Luenberger/Kalman-like linear observers for nonlinear estimation. Observers designed based on KOF converge under an appropriate observability condition, which can be computed in terms of specific Koopman eigenvalues and Koopman modes used for constructing KOF: thus for first time establishing connection between Koopman spectral properties and observer synthesis. Using the techniques for Koopman eigenvalue/eigenfunction/mode computation, we describe a numerical procedure to transform a given nonlinear system into KOF which is often valid in a large portion of state space (e.g. basin of attraction) or even globally. We numerically compare our

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approach with the EKF and report superior estimation performance.

The paper is organized into three main sections. In Section 2 we review key concepts from the Koopman operator theory. A theoretical and computational framework for Koopman operator based observer synthesis is introduced in Section 3. We numerically illustrate this framework for nonlinear estimation in Section 4. Finally, we conclude in Section 5 with directions for future research.

2. OVERVIEW OF KOOPMAN OPERATOR THEORETIC TECHNIQUE

In this section we briefly overview Koopman operator theoretic concepts, see Mezic (2012) and Budisic et al. (2012) for details. Consider an autonomous discrete time nonlinear dynamical system

$$\mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}), \tag{1}$$

where, $\mathbf{x}_t \in \mathbf{X} \subset \mathbb{R}^d$ is a state vector, $\mathbf{f} : \mathbf{X} \to \mathbf{X}$ is a function which describes the nonlinear state evolution. Let $\mathcal{F}$ be space of *observables* which are scalar-valued functions $\theta : \mathbf{X} \to \mathbb{C}$ (where, $\mathbb{C}$ denotes the complex plane) defined on the state space. We assume that the observables are continuous, i.e. $\mathcal{F} \subset \mathcal{C}^0(\mathbf{X})$ (which is space of all continuous functions on $\mathbf{X}$), see Mauroy and Mezic (Unpublished) for further discussion on choice of $\mathcal{F}$. The Koopman operator is a linear operator $\mathcal{U} : \mathcal{F} \to \mathcal{F}$ which maps θ into a new function $\mathcal{U}\theta$, as follows

$$(\mathcal{U}\theta)(\mathbf{x}) = \theta(\mathbf{f}(\mathbf{x})). \tag{2}$$

Although the dynamical system (1) is nonlinear and evolves on a finite-dimensional space, the Koopman operator $\mathcal{U}$ is linear but infinite-dimensional. The eigenvalues λ of Koopman operator, referred to as *Koopman eigenvalues* (KEs), and the eigenfunctions ϕ of Koopman operator, referred to as *Koopman eigenfunctions* (KEFs) are defined as follows:

$$\mathcal{U}\phi = \lambda\phi. \tag{3}$$

The set of all Koopman eigenvalues $\lambda_j, j = 1, 2, \cdots$ is called the point spectrum of the Koopman operator (Mezic (2005)). Koopman operator may also have residual and continuous parts of spectrum, but for the purpose of this paper, the point spectrum will suffice. Note that if ϕ_1, ϕ_2 are Koopman eigenfunctions with eigenvalues λ_1, λ_2, then $\phi_1\phi_2$ is also an eigenfunction with eigenvalue $\lambda_1\lambda_2$. Using the relation

$$\phi(\mathbf{x}_t) = \phi(\mathbf{f}(\mathbf{x}_{t-1})) = \mathcal{U}\phi(\mathbf{x}_{t-1}) = \lambda\phi(\mathbf{x}_{t-1}), \tag{4}$$

it follows

$$\phi(\mathbf{x}_t) = \lambda^t\phi(\mathbf{x}_0). \tag{5}$$

Let $\mathbf{g} : \mathbf{X} \to \mathbb{R}^m$ be a vector valued observable. If each of the m components of $\mathbf{g}$ lie within the span of eigenfunctions $\phi_j, j = 1, 2\cdots$, then $\mathbf{g}$ can be expanded in terms of these eigenfunctions as (see Mezic (2005)),

$$\mathbf{g}(\mathbf{x}) = \sum_{j=1}^{\infty} \phi_j(\mathbf{x})\mathbf{v}_j, \tag{6}$$

where, $\mathbf{v}_j \in \mathbb{C}^m$ are complex valued vectors. Using (5), the time evolution $\mathbf{g}(\mathbf{x}_t)$ can be expressed as

$$\mathbf{g}(\mathbf{x}_t) = \sum_{j=1}^{\infty} \lambda_j^t \phi_j(\mathbf{x}_0)\mathbf{v}_j. \tag{7}$$

We will refer to this expansion as Koopman Mode Decomposition (KMD) following (Susuki and Mezic (2014)), with $\mathbf{v}_j$ being the Koopman modes associated with eigenfunction ϕ_j and the observable $\mathbf{g}$. The modes capture correlations in components of the observable, while the corresponding eigenvalues define growth/decay rates and oscillation frequencies for the mode. If the dynamics have only a finite number of discrete spectra (peaks) in complex plane, then a finite truncation of expansion (7) gives a good approximation of the dynamics. KMD can be thought of as a generalized Fourier analysis, and offers several advantages over Discrete Fourier Transform, see Chen et al. (2012). Each Koopman mode represents only one frequency component, and thus is expected to decouple dynamics at different time scales more effectively than Proper Orthogonal Decomposition (Susuki et al. (2011)).

Note that while KEs/KEFs are intrinsic to the dynamics (1), the modes depend on the choice of the observable, i.e. $\mathbf{g}$. We will refer to modes $\mathbf{v}^x$ for full state observable, i.e. $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ as the *Koopman Modes* (KMs) and modes $\mathbf{v}^{\mathbf{h}}$ for any other observable $\mathbf{h}$ as the *Output Koopman Modes* (OKMs). Finally, we will refer to the KEs, KEFs, KMs triplet i.e. $(\lambda_i, \phi_i, \mathbf{v}_i^x), i = 1, \cdots$ as the *Koopman tuple*.

Computation of Koopman tuple is a challenging problem and is an active area of research. A variety of techniques have been proposed in literature, including harmonic averaging (Mezic (2005); Mezic and Banaszuk (2004)), generalized Laplacian analysis (Budisic et al. (2012)), and Dynamic Mode Decomposition (DMD) and its variants, (see Tu et al. (2014) and references there in), and extended DMD (Williams et al. (2015a)). These approaches are equation free (for a technique which explicitly uses the equations, see Mauroy and Mezic (Unpublished)) and rely on time traces/snapshots of appropriate observables generated from the system (1) . In our application we will rely on extended DMD which we discuss in detail in Sec. 3.4.

3. KOOPMAN OPERATOR BASED OBSERVER SYNTHESIS

Consider a discrete time nonlinear system with outputs

$$\begin{aligned}\mathbf{x}_t &= \mathbf{f}(\mathbf{x}_{t-1}),\\ \mathbf{y}_t &= \mathbf{h}(\mathbf{x}_t),\end{aligned} \tag{8}$$

where, $\mathbf{h} : \mathbf{X} \to \mathbb{R}^m$ is the output function or observable on the state space $\mathbf{X}$. An observer (often designed as an auxiliary dynamical system), is a casual mapping from any prior information about the initial condition $\mathbf{x}_0$ and the past outputs $\{\mathbf{y}_t : t_0 \le t \le t\}$ to an estimate $\hat{\mathbf{x}}_t$ of the current state.

3.1 Nonlinear Observers

To put the Koopman operator based observer synthesis framework in context of existing literature, we first briefly review observer design techniques based on *observer forms*. Such techniques include global/pseudo/extended linearization, linearization by input-output injection or error linearization, immersion based design, etc., see Nijmeijer and Fossen (1999); Besancon (2007); Keller (1987); Kang et al. (2013); Unbehauen (2009) and references therein. The key idea behind such techniques is to seek a transformation

$$\mathbf{u} = \mathcal{V}(\mathbf{x}), \qquad \mathbf{w} = \mathcal{W}(\mathbf{y}), \tag{9}$$

such that (8) can be converted into a canonical observer form, e.g.

$$\mathbf{u}_t = A^{co}\mathbf{u}_{t-1} + \alpha(\mathbf{y}_t), \tag{10}$$

$$\mathbf{w}_t = C^{co}\mathbf{u}_t + \beta(\mathbf{y}_t), \tag{11}$$

where, $\alpha(\mathbf{y}), \beta(\mathbf{y})$ are referred to as output injection terms. One can then design a Luenberger-like linear observer,

$$\hat{\mathbf{u}}_t = A^{co}\hat{\mathbf{u}}_{t-1} + \alpha(\mathbf{y}_t) + L(\mathbf{w}_t - \hat{\mathbf{w}}_t), \quad (12)$$

$$\hat{\mathbf{w}}_t = C^{co}\hat{\mathbf{u}}_t + \beta(\mathbf{y}_t), \quad (13)$$

$$\mathbf{x}_t = \mathcal{V}^{-1}(\hat{\mathbf{u}}_t), \quad (14)$$

where, L is the Luenberger gain. The error $\mathbf{e}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$ follows a linear dynamics: $\mathbf{e}_t = A^{co}\mathbf{e}_t$. Thus, if (A^{co}, C^{co}) is observable, then L can be chosen so that $A^{co} - LC^{co}$ is Hurwitz. The transformations (9) must satisfy a first order system of partial differential equations (PDEs) and must atleast be local diffeomorphisms. To be solvable the system of PDEs must satisfy geometric integrability conditions (Krener and Respondek (1985)). A more general observer form is the so-called state affine form in which $A^{co}(\mathbf{y})$ and $C^{co}(\mathbf{y})$ are functions of outputs. For this form one has to employ Kalman-like observer instead of the Luenberger approach. The PDEs for resulting transformations are more complicated, but the integrability conditions are less stringent (Unbehauen (2009)). Along similar lines, immersion based approaches have been proposed (see (Besancon (2007)) and references there in) which do not restrict the state transformation to be a diffemorphism, and employ immersion of state space into a space of larger dimension to facilitate observer synthesis.

The problem with above approaches is that there are often restrictive, i.e. only very few systems in practise can be transformed into the desired observer forms. These techniques have been at best employed for low dimensional problems, and the computations grow in complexity as the state dimension increases. We next discuss how Koopman operator theoretic framework can provide a practical approach for obtaining a suitable observer form which potentially makes linear observers accessible in a wide range of nonlinear estimation problems.

3.2 Koopman Observer Form

Assumption I: Let $\mathcal{F}^n = \text{span}\{\phi_i\}_{i=1}^n$ be a subset of KEFs for system (8) such that $\mathbf{h}(\mathbf{x}), \mathbf{x} \in \mathcal{F}^n$, and so

$$\mathbf{x} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i^x, \quad \mathbf{h}(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{h}}, \quad (15)$$

where, $\mathbf{v}_i^x \in \mathbb{C}^d, i = 1, \cdots, n$ are the KMs, and $\mathbf{v}_i^{\mathbf{h}} \in \mathbb{C}^m, i = 1, \cdots, n$ are the OKMs as defined before.

Note that if λ is a complex KE with KEF ϕ, then the complex conjugate $\overline{\lambda}$ is also a KE with KEF $\overline{\phi}$. Similarly, for real valued observables $\mathbf{h}$, the KMs occur in conjugate pairs. In whatever follows, we order KEFs $\{\phi_1, \phi_2, \cdots, \phi_n\}$ (and correspondingly KEs and KMs/OKMs) such that complex conjugate appears adjacent to each other. We shall denote by: $\text{Re}(c)$, $\text{Im}(c)$, $|c|$ and $\arg(c)$ as the real part, imaginary part, modulus and argument, respectively of any complex number $c \in \mathbb{C}$; and by $*$ as the standard vector/matrix transpose.

Define $\mathcal{T}_{\mathcal{U}}(\mathbf{x}) = (\hat{\phi}_1(\mathbf{x}), \hat{\phi}_2(\mathbf{x}), \cdots, \hat{\phi}_n(\mathbf{x}))^*$ as follows:

- $\hat{\phi}_i = \phi_i$ if i-th KEF is real, and
- $\hat{\phi}_i = 2\text{Re}(\phi_i)$ and $\hat{\phi}_{i+1} = -2\text{Im}(\phi_i)$, if i and $i+1$-th KEFs are complex conjugate pairs.

Consider a nonlinear change of coordinates defined by $\mathcal{T}_{\mathcal{U}} : \mathbb{R}^d \to \mathbb{R}^n$,

$$\mathbf{z}_t = \mathcal{T}_{\mathcal{U}}(\mathbf{x}_t). \quad (16)$$

We refer to this transformation as the *Koopman Canonical Transform* (KCT), and the coordinates $\mathbf{z}_t = (\mathbf{z}_{1,t}, \cdots, \mathbf{z}_{n,t})^* \in \mathbb{R}^n$ as the *Koopman Canonical Coordinates* (KCC). From Eqn. (4), it follows:

- $\mathbf{z}_{i,t} = \lambda_i \mathbf{z}_{i,t-1}$, if i-th KEF is real,
- $(\mathbf{z}_{i,t}, \mathbf{z}_{i+1,t})^* = Q_{\lambda_i}(\mathbf{z}_{i,t-1}, \mathbf{z}_{i+1,t-1})^*$, if i and $i+1$-th KEFs are complex conjugate pairs, where,
$$Q_\lambda = |\lambda| \begin{pmatrix} \cos(\arg\lambda) & \sin(\arg\lambda) \\ -\sin(\arg\lambda) & \cos(\arg\lambda) \end{pmatrix}. \quad (17)$$

It follows then, $\mathbf{z}_t = A\mathbf{z}_{t-1}$ where, A is a $n \times n$ real block diagonal matrix such that:

- A has a diagonal entry $A_{i,i} = \lambda_i$, if i-th KEF is real,
- A has a block diagonal entry $\begin{bmatrix} A_{i,i} & A_{i,i+1} \\ A_{i+1,i} & A_{i+1,i+1} \end{bmatrix} = Q_{\lambda_i}$, if i and $i+1$-th KEFs are complex conjugate pairs.

Furthermore, it is straightforward to show that KMD (15) can be expressed in terms of KCC as

$$\mathbf{x}_t = C^x\mathbf{z}_t, \quad \mathbf{h}(\mathbf{x}_t) = C^{\mathbf{h}}\mathbf{z}_t, \quad (18)$$

where, $C^x \in \mathbb{R}^{d\times n}$ and $C^{\mathbf{h}} \in \mathbb{R}^{m\times n}$ are matrices obtained from KMs and OKMs, respectively. For instance, i-th column of C^x is $\mathbf{v}_i^x$ if i-th KEF is real, and $i, i+1$-th columns are $\text{Re}(\mathbf{v}_i)$ and $\text{Im}(\mathbf{v}_i)$, respectively if i and $i+1$-th KEFs are complex conjugate pairs. Similar construction applies for $C^{\mathbf{h}}$.

In summary, using KCC $\mathbf{z}_t$, the evolution of full state observable $\mathbf{x}_t$ and $\mathbf{h}(\mathbf{x}_t)$ can be expressed via a linear time invariant system with outputs:

$$\mathbf{z}_t = A\mathbf{z}_{t-1}, \quad (19)$$

$$\mathbf{h}(\mathbf{x}_t) = C^{\mathbf{h}}\mathbf{z}_t, \quad (20)$$

$$\mathbf{x}_t = C^x\mathbf{z}_t. \quad (21)$$

We will refer to the system (19-21) as the *Koopman Observer Form* (KOF). Some remarks follow:

- Following as similar procedure as outlined above, one can derive an analogous KOF in continuous time setting.
- KOF does not explicitly depend on KEFs.
- Compared to the observer form (10-11), in the KOF there are no output injection terms.
- In the observer (12-14) based on standard observer form, the state estimate is obtained by the inverse nonlinear transform (see Eqn. (14)). In contrast, in KOF, state estimate can be obtained via a linear transform (see Eqn. (21)), thereby providing a computational advantage.
- Typically $n \gg d$, i.e the dimension of KOF can be significantly greater than the state space dimension of the original nonlinear system. In this regard Koopman approach can be thought of as an immersion based design, see Besancon (2007). When $n \to \infty$, a finite approximation $n = n_o \gg 1$ has to be introduced from a practical perspective. Different approaches for approximations and analysis of approximation error is beyond the scope of this paper, and will be investigated in future work. The infinite dimensional linear systems theory (Temam (1997)) could play an important role in such an analysis.
- Finally, note that in above construction of KOF we have assumed KEFs to be simple. One can extend the above construction for generalized KEFs (see Budisic et al. (2012)) as well, in that case A will have a more general block diagonal structure.

Given the KOF, one can design a standard Luenberger observer (or Kalman filter, see Section 3.5 for further discussion)

$$\hat{\mathbf{u}}_t = A\hat{\mathbf{u}}_{t-1} + L(\mathbf{y}_t - \hat{\mathbf{y}}_t), \quad (22)$$

$$\hat{\mathbf{y}}_t = C^{\mathbf{h}}\hat{\mathbf{u}}_t, \quad (23)$$

$$\mathbf{x}_t = C^x\hat{\mathbf{u}}_t. \quad (24)$$

As before if the pair $(A, C^{\mathbf{h}})$ is observable then the observer gain L can be chosen such that the observer (22-24) converges.

Finally, note that if an exact finite dimensional KMD (see Eqn. (15)) holds, then KOF is globally valid and observer designed based on it will be globally convergent under observability conditions stated above. For what classes of systems one can achieve this will be investigated in future. It is clear, however that since KOF can only have one equilibria at origin, an exact finite dimensional KMD/KOF can only be possible if system (8) has only one equilibria. When system (8) has multiple equilibria or attractors in general, it may be possible to get a finite KOF restricted only to basins of attraction. In addition there may be cases where infinite many terms are needed in KMD, i.e. $n \to \infty$ in Eqn. (15), and one will have to introduce a truncation (as pointed above), and thus the KCT may be accurate only locally in a subset of the state space $\mathbf{X}$. In those cases the observer design based on the KOF may only exhibit local convergence. Still, this convergence is expected to hold over a much larger potion of state space (e.g. basin of attraction) compared to the EKF, for example.

While additional theoretical work is required to characterize tradeoff between accuracy and size of KOF, and region of validity of KOF, we numerically demonstrate that Koopman framework could provide a practical and superior approach for observer design. In addition availability of scalable numerical techniques for Koopman tuple computation (as discussed in Section 3.4) makes this approach viable for a large class of nonlinear systems including high dimensional ones.

3.3 Nonlinear Observability

We will use standard notion of nonlinear observability Hermann and Krener (1977), which we briefly recall first. A pair of points $\mathbf{x}_0$ and $\mathbf{x}_0'$ are *indistinguishable* (denoted $\mathbf{x}_0 \mathbb{I} \mathbf{x}_0'$) if the system (8) with these two initial conditions realizes same sequence of outputs $\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \cdots\}$. Consequently, system (8) is said to be *nonlinearly observable at* $\mathbf{x}_0$ if $\mathbb{I}(\mathbf{x}_0) = \{\mathbf{x}_0\}$ and is *nonlinearly observable* if $\mathbb{I}(\mathbf{x}) = \{\mathbf{x}\}$ for every $\mathbf{x} \in \mathbf{X}$.

Theorem I: If Assumption I (see Section 3.2) holds and the pair $(A, C^{\mathbf{h}})$ for the KOF (19-21) is observable, then system (8) is nonlinearly observable, and L can be chosen such that the linear observer (22-24) converges.

Proof: The proof is by contradiction. First, note that since $\mathbf{x} \equiv C^x \mathbf{z}$, the transformation $\mathcal{T}_{\mathcal{U}}$ is an injective map. Next assume that the system (8) is not observable, and so there exists two distinct initial conditions $\mathbf{x}_0 \neq \mathbf{x}_0'$ resulting in same output sequence $\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \cdots\}$. Let $\mathbf{z}_0 = \mathcal{T}_{\mathcal{U}}(\mathbf{x}_0)$ and $\mathbf{z}_0' = \mathcal{T}_{\mathcal{U}}(\mathbf{x}_0')$, then $\mathbf{z}_0 \neq \mathbf{z}_0'$ by injectivity of $\mathcal{T}_{\mathcal{U}}$. By construction, the KOF (19-21) will also produce same output sequence starting at $\mathbf{z}_0$ and $\mathbf{z}_0'$. This is a contradiction, since the KOF is observable under conditions of *Theorem I*, and so $\mathbf{z}_0 = \mathbf{z}_0'$.

It is interesting to note how certain KEs (forming A) and OKMs (forming $C^{\mathbf{h}}$) play a role in determining nonlinear observability and convergence of observer design.

3.4 Koopman Tuple Computation

In this section we summarize the extended DMD (EDMD) approach (Williams et al. (2015a)) and its kernel version (Williams et al. (Unpublished)) which we will use for Koopman tuple computation for obtaining the KOF. EDMD is a extension of DMD which has emerged as powerful tool for analyzing nonlinear systems. DMD originally introduced in fluids community (Schmid (2010)), characterizes the nonlinear dynamics through analysis of some approximating linear systems. It uses Arnoldi type methods for computing DMD eigenvalues/modes as empirical Ritz eigenvalues/vectors from sequential data. It was shown in (Rowley et al. (2009)) that DMD is closely related to KMD, and under certain conditions DMD eigenvalues/modes are equivalent to KEs/KMs. This connection has been further strengthened in exact DMD approach (Tu et al. (2014)), which generalizes the notion of approximating linear system used in DMD, and is able to handle non-sequential data. EDMD further extends exact DMD in the sense that it reduces to exact DMD under a specific choice of dictionary used in EDMD.

EDMD is a Galerkin weighted residual approach which uses a dictionary of basis functions to approximate KEFs and corresponding KEs. Let $\mathcal{F}_D \subset \mathcal{F}$ be a subset of observables spanned by a dictionary $\mathcal{D} \equiv \{\psi_1, \cdots, \psi_D\}$, where $\psi_i : \mathbf{X} \to \mathbb{C}$. Then $\theta, \hat{\theta} \in \mathcal{F}_D$ can be expressed as $\theta(\mathbf{x}) = \Psi^*(\mathbf{x})\mathbf{a}$, $\hat{\theta}(\mathbf{x}) = \Psi^*(\mathbf{x})\hat{\mathbf{a}}$, respectively, for some $\mathbf{a}, \hat{\mathbf{a}} \in \mathbb{C}^D$, where $\Psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \cdots, \psi_D(\mathbf{x}))^*$. Under the action of Koopman operator

$$\begin{aligned} \mathcal{U}(\theta)(\mathbf{x}) &= (\Psi \circ \mathbf{f}(\mathbf{x}))^* \mathbf{a} = \Psi^*(\mathbf{x})\hat{\mathbf{a}} + r(\mathbf{x}) \\ &= \Psi^*(\mathbf{x}) U \mathbf{a} + r(\mathbf{x}), \end{aligned} \tag{25}$$

where, U is a finite dimensional approximation of $\mathcal{U}$, and $r(\mathbf{x})$ is the residual as $\mathcal{F}_D$ may not be invariant under action of $\mathcal{U}$. Given a dataset of snapshot pairs $\{(\mathbf{x}_i, \overline{\mathbf{x}}_i)\}_{i=1}^N$, $\overline{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$, one can formulate a least square problem of minimizing,

$$\sum_{i=1}^{N} |r(\mathbf{x}_i)|^2 = \sum_{i=1}^{N} | \left(\Psi^*(\overline{\mathbf{x}}_i) - \Psi^*(\mathbf{x}_i) U\right) \mathbf{a}|,$$

to obtain

$$U = \mathbf{\Psi}_{\mathbf{x}}^{\dagger} \mathbf{\Psi}_{\overline{\mathbf{x}}}, \tag{26}$$

where, $\dagger$ is the pseudo inverse and

$$\mathbf{\Psi}_{\mathbf{x}} = \begin{bmatrix} \Psi^*(\mathbf{x}_1) \\ \Psi^*(\mathbf{x}_2) \\ \vdots \\ \Psi^*(\mathbf{x}_N) \end{bmatrix}_{N \times D}, \quad \mathbf{\Psi}_{\overline{\mathbf{x}}} = \begin{bmatrix} \Psi^*(\overline{\mathbf{x}}_1) \\ \Psi^*(\overline{\mathbf{x}}_2) \\ \vdots \\ \Psi^*(\overline{\mathbf{x}}_N) \end{bmatrix}_{N \times D}.$$

Let $\lambda_i, i = 1, \cdots, D$ be eigenvalues of U, with corresponding right/left eigenvectors ξ_i, γ_i, respectively. Then λ_i approximate KEs with corresponding KEFs given by $\phi_i(\mathbf{x}) = \Psi^*(\mathbf{x})\xi_i$. Let the coordinate function $g_i(\mathbf{x}) = x_i$ be in span of $\mathcal{D}$ so that $x_i = \Psi^* \mathbf{b}_i$ for some $\mathbf{b}_i \in \mathbb{R}^D$. Then, KMs can be obtained via $\mathbf{v}_i = B^* \gamma_i$, where $B = [\mathbf{b}_1, \cdots \mathbf{b}_n]$. Similar computations result in OKMs.

EDMD approach discussed above suffers from curse of dimensionality due to explosion in number of required dictionary elements D with the increase in state dimension d. To circumvent explicit construction of the dictionary, a kernel based EDMD approach has been proposed in Williams et al. (Unpublished). In this approach the computation of eigenvectors/eigenvalues of U is accomplished by forming an alternative matrix

$$\hat{U} = \hat{G}^{\dagger} \hat{A}, \tag{27}$$

where, $\hat{G} = \mathbf{\Psi}_{\mathbf{x}} \mathbf{\Psi}_{\mathbf{x}}^*$ and $\hat{A} = \mathbf{\Psi}_{\overline{\mathbf{x}}} \mathbf{\Psi}_{\mathbf{x}}^*$ are $N \times N$ matrices. Using the kernel trick, entries of matrices $\hat{G}, \hat{A}$ can be computed directly (without forming $\Psi(\mathbf{x})$ for computing inner products of form $\Psi^*(\mathbf{x}_i)\Psi(\mathbf{x}_j)$, an $\mathcal{O}(D)$ operation) as

$$\hat{G}_{ij} = K(\mathbf{x}_i, \mathbf{x}_j), \quad \hat{A}_{ij} = K(\overline{\mathbf{x}}_i, \mathbf{x}_j), \tag{28}$$

where, $K(\mathbf{x}, \mathbf{x}) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ is an appropriately chosen kernel function. K implicitly defines $\mathcal{F}_D$ (subspace of scalar observables spanned by elements of $\Psi(\mathbf{x})$), and evaluates the inner products implicity in $\mathcal{O}(d)$ rather than $\mathcal{O}(D)$ time. The steps for Koopman tuple computation based on this approach are summarized in Algo 1. The total computational cost of this approach is $\mathcal{O}(N^2 \max(d, N))$. For problems with large number of snapshots N, Krylov methods could be used to compute a (leading) subset of the eigenvalues/vectors.

Algorithm 1 Kernel based EDMD

1: Input: Dataset with snapshot pairs $\{(\mathbf{x}_i, \overline{\mathbf{x}}_i)\}_{i=1}^N$, kernel function K and observable $\mathbf{h}$
2: Output: Koopman tuple $(\{(\lambda_i, \phi_i, \mathbf{v}_i^x)\}_{i=1}^N$ and OKMs $\{\mathbf{v}_i^{\mathbf{h}}\}_{i=1}^N$
3: Compute $\hat{U}$ by forming $\hat{G}, \hat{A}$ using (28).
4: Compute eigenvalues $\hat{\lambda}_i, i = 1, \cdots, N$, and corresponding right eigenvectors $\hat{\xi}_i$ of $\hat{U}$.
5: Let $\hat{\Xi} = [\hat{\xi}_1 \cdots \hat{\xi}_N]$, then i-th rows of

$$\Phi_{\mathbf{x}} = \hat{G}\hat{\Xi}, \qquad \Phi_{\overline{\mathbf{x}}} = \hat{A}\hat{\Xi}, \tag{29}$$

contains the numerically computed KEFs evaluated at $\mathbf{x}_i$ and $\overline{\mathbf{x}}_i$, respectively.
6: Compute KMs/OKMs as

$$\begin{aligned} [\mathbf{v}_1^x, \cdots, \mathbf{v}_N^x] &= (\Phi_{\mathbf{x}})^\dagger X, \\ [\mathbf{v}_1^{\mathbf{h}}, \cdots, \mathbf{v}_N^{\mathbf{h}}] &= (\Phi_{\mathbf{x}})^\dagger H_{\mathbf{x}}, \end{aligned} \tag{30}$$

where, $X = [\mathbf{x}_1, \cdots, \mathbf{x}_N]^*$ and $H_{\mathbf{x}} = [\mathbf{h}(\mathbf{x}_1), \cdots, \mathbf{h}(\mathbf{x}_N)]^*$.

Some remarks follow:

- Choice of dictionary/kernel function: Optimal choice of $\mathcal{D}$ or equivalently K is an important, yet open question. These choices will most likely depend both on underlying dynamical system and strategy used to obtain the dataset. Examples of $\mathcal{D}$ include polynomials, spectral elements, radial basis functions etc. (Williams et al. (2015a)), while choices of kernels include polynomial, Gaussian, Matern, etc. see Rasmussen and Williams (2006). Note that when $\mathcal{F}_{\mathcal{D}}$ is not an invariant subspace of $\mathcal{U}$, than it can result in errors in computation of some eigenfunctions. Further, since KEFs are discontinuous across basins of attraction, as appropriate choice of dictionary/kernel is critical to obtain approximation over larger domain which extend beyond basin of attraction. We refer the reader to see Williams et al. (2015a); Mauroy and Mezic (Unpublished) for further discussions on such issues.
- Choice of sampling points: EDMD procedure being a weighted residual Galerkin method converges as $N \to \infty$. With randomly distributed samples, the convergence rate behaves as $\mathcal{O}(N^{-1/2})$ as in Monte Carlo integration techniques. Other sampling choices, e.g. uniform grid, effectively uses different quadrature rules and could lead to better convergence rates.

3.5 Computation of KOF and Observer Design

Algo. 2 summarizes the procedure for constructing KOF. In Step 4, any approach for Koopman tuple computation (as discussed in Sections 3.4/2) can be used. Note that for a given system, KOF computation can be carried out once in an off-line fashion.

Algorithm 2 Procedure for obtaining KOF

1: Input: $\mathbf{f}$, $\mathbf{h}$
2: Output: KOF $(A, C^x, C^{\mathbf{h}})$
3: Generate dataset with snapshot pairs $\{(\mathbf{x}_i, \overline{\mathbf{x}}_i)\}_{i=1}^N$, where $\overline{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$ by simulating system (1).
4: Compute Koopman tuple $(\lambda_i, \phi_i, \mathbf{v}_i^x)$ and OKMs $\mathbf{v}_i^{\mathbf{h}}$, $i = 1, \cdots, n$, e.g. using Algo (1).
5: Use the procedure described in Section 3.2 to obtain the KOF $(A, C^x, C^{\mathbf{h}})$.

Once KOF is available, one can use any linear observer, e.g. Luenberger or Kalman filter, for estimation as discussed in Section 3.1. An additional step is required for initializing the observer $\mathbf{z}_0$ based on initial state specifications in the original coordinates $\mathbf{x}_0$. Specifically,

- *Initial state*: $\mathbf{z}_0$ can be obtained directly by using the KCT (16), i.e. $\mathbf{z}_0 = \mathcal{T}_{\mathcal{U}}(\mathbf{x}_0)$. One could also invert the relation (20) to approximately obtain $\mathbf{z}_0 \approx (C^{\mathbf{h}})^\dagger \mathbf{x}_0$.
- *Initial distribution*: For the case when a distribution for $\mathbf{x}_0$ is prescribed, e.g. $\mathbf{x}_0 \sim \mathcal{N}(\overline{\mathbf{x}}_0, P_0^x)$ (where, $\mathcal{N}(\mathbf{x}, P)$ denotes a normal distribution with mean $\mathbf{x}$ and covariance P), one has to resort to a numerical procedure to obtain corresponding distribution on $\mathbf{z}_0$. Note that even when $\mathbf{x}_0$ is normally distributed, $\mathbf{z}_0$ will in general not be normally distributed under KCT. Due to non-normally distributed initial condition, the distribution on $\mathbf{z}_t$ (and similarly for $\mathbf{x}_t$ via relation (21)) will evolve as a non-normal distribution even though KOF is linear, which is no surprise (as under nonlinear evolution (1), a normally distributed $\mathbf{x}_t$ at $t = 0$ could evolve into a general distribution). In our application we use the Kalman filter with the approximation $\mathbf{z}_0 \sim \mathcal{N}(\overline{\mathbf{z}}_0, P_0^{\mathbf{z}})$ where,

$$\overline{\mathbf{z}}_0 \equiv (C^{\mathbf{h}})^\dagger \overline{\mathbf{x}}_0, \quad P_0^{\mathbf{z}} \equiv (C^{\mathbf{h}})^\dagger P_0^x ((C^{\mathbf{h}})^\dagger)^*. \tag{31}$$

Finally, note that once the filter estimates in KCC $\{\hat{\mathbf{z}}_t\}_{t=1}^T$ have been obtained from a given output sequence $\{\mathbf{y}_t\}_{t=1}^T$, the corresponding state estimates $\{\hat{\mathbf{x}}_t\}_{t=1}^T$ can be obtained using $\hat{\mathbf{x}}_t = C^x \hat{\mathbf{z}}_t$. One can recover approximate state error covariance $\hat{P}_t^x = C^x \hat{P}_t^{\mathbf{z}} (C^x)^*$ from $\{\hat{P}_t^{\mathbf{z}}\}_{t=1}^T$ if one uses the Kalman filter.

4. NUMERICAL DEMONSTRATION

In this section we numerically demonstrate Koopman operator framework for observer synthesis. We consider discrete time nonlinear system with output noise

$$\begin{aligned} \mathbf{x}_t &= \mathbf{f}(\mathbf{x}_{t-1}), \\ \mathbf{y}_t &= \mathbf{h}(\mathbf{x}_t) + \mathbf{s}_t, \end{aligned} \tag{32}$$

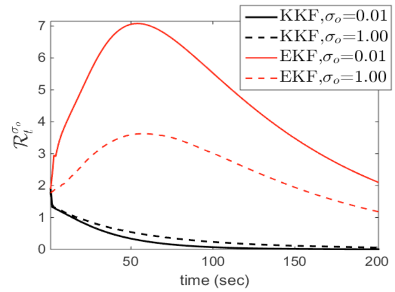
where, $\mathbf{s}_t \sim \mathcal{N}(\mathbf{0}, R)$ is zero mean normally distributed noise with covariance matrix R. Also, we assume that the initial condition is uncertain with normal distribution $\mathbf{x}_0 \sim \mathcal{N}(\overline{\mathbf{x}}_0, P_0^x)$. In what follows we consider output noise covariance to be of the form $R = \sigma_o^2 \mathcal{I}_m$, where $\sigma_o > 0$ and similarly initial state covariance to be form $P_0^x = \sigma_s^2 \mathcal{I}_d$, where $\mathcal{I}_p$ denotes a $p \times p$ identity matrix. Note that additive noise term in (32), leads to a similar additive term in output Eqn. (20) of the KOF. In all the examples considered, we use Kalman filter based on KOF for estimation, and for brevity we will refer to that filter as the Koopman Kalman Filter (KKF). For comparison we use the EKF with initial condition sampled according to $\mathbf{x}_0 \sim \mathcal{N}(\overline{\mathbf{x}}_0, P_0^x)$. The KKF is initialized consistently with this initial condition using the procedure (see Eqn. (31)) described in Section 3.5.

To assess estimation performance, we obtain several realizations of noisy outputs $\{\mathbf{y}_1^i, \cdots, \mathbf{y}_T^i\}_{i=1}^{N_r}$ by simulating the system (32) with random initial conditions $\mathbf{x}_0^i \sim \mathcal{N}(\overline{\mathbf{x}}_0, P_0^x), i = 1 \cdots, N_r$ at a given noise levels σ_o. Let $\{\mathbf{x}_0^i, \cdots, \mathbf{x}_T^i\}_{i=1}^{N_r}$ be underlying state sequence and let $\{\hat{\mathbf{x}}_0^i, \cdots, \hat{\mathbf{x}}_T^i\}_{i=1}^{N_r}$ be its estimate, then estimation accuracy can be measured using root mean square error as a function of time:

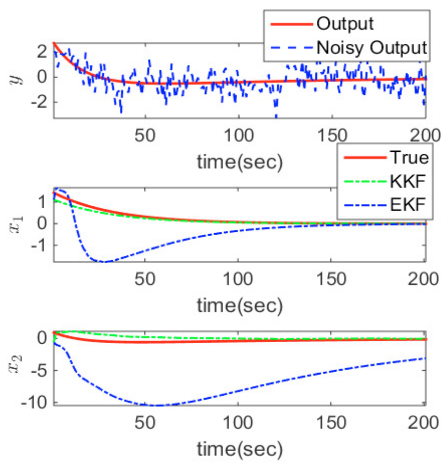
$$\mathcal{R}_t^{\sigma_o} = \sqrt{\frac{1}{N_r} \sum_{i=1}^{N_r} ||\mathbf{x}_t^i - \hat{\mathbf{x}}_t^i||^2}, \quad t = 1, \cdots, T. \tag{33}$$

4.1 Example I

Consider a nonlinear map f in Eqn. (32) to be

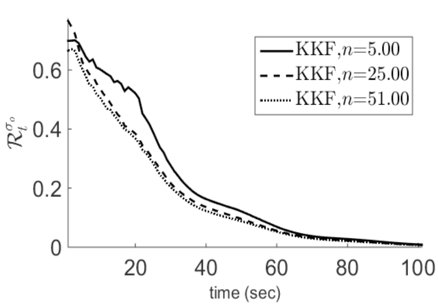


(a)



(b)

Fig. 1. a) $\mathcal{R}_t^{\sigma_o}$ for EKF and KKF for for $\sigma_o = 0.01$ and $\sigma_o = 1$ for the system (34). b) Responses of KKF/EKF for one of the realizations of the output with $\sigma_o = 1$.

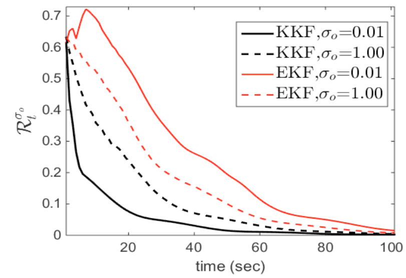


(a)

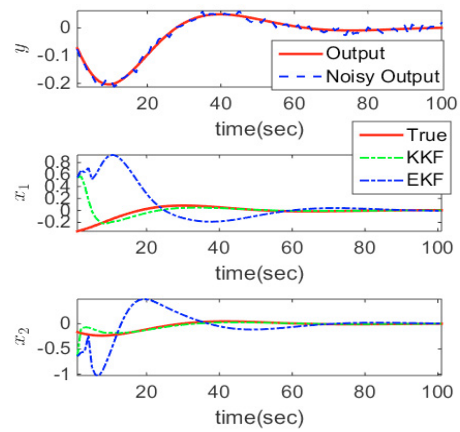
Fig. 2. $\mathcal{R}_t^{\sigma_o}(\sigma_o = 1)$ for KKF based on different order n of KOF for the system (36).

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \rho x_1 \\ \mu x_2 + (\rho^2 - \mu)c x_1^2 \end{pmatrix}, \tag{34}$$

and let $\mathbf{h}(\mathbf{x}) = x_1^2 + x_2$ be the output. The system has a globally stable fixed point at the origin. It can be shown that for (34), ρ, μ are Koopman eigenvalues with eigenfunctions $\phi_\rho(\mathbf{x}) = x_1$, and $\phi_\mu(\mathbf{x}) = x_2 - c x_1^2$, respectively. Also note that $\rho^2, \rho\mu$ etc. are Koopman eigenvalues with eigenfunctions $\phi_\rho^2, \phi_\rho\phi_\mu$ etc. It follows that $\mathbf{x} = \sum_{i=1}^3 \phi_i(\mathbf{x})\mathbf{v}_i$ where, $\phi_1 = \phi_\rho, \quad \phi_2 = \phi_\mu, \quad \phi_3 = \phi_\rho^2$ and $\mathbf{v}_1 = (1,0)^*, \mathbf{v}_2 = (0,1)^*$ and $\mathbf{v}_3 = (0,c)^*$ are the KMs. Similarly, $\mathbf{h}_2(\mathbf{x}) = \phi_2(\mathbf{x}) + (c+1)\phi_3(\mathbf{x})$ defining the OKMs. Thus, using the KCT



(a)



(b)

Fig. 3. a) $\mathcal{R}_t^{\sigma_o}$ for EKF and KKF for $\sigma_o = 0.01$ and $\sigma_o = 1$ for the system (36). b) Responses of KKF/EKF for one of the realizations of the output with $\sigma_o = 0.01$.

$$\mathbf{z} = \mathcal{T}_{\mathcal{U}}(\mathbf{x}) = \begin{pmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \\ \phi_3(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 - c\mathbf{x}_1^2 \\ \mathbf{x}_1^2 \end{pmatrix}, \tag{35}$$

one obtains KOF (19-21) where, $A = \operatorname{diag}(\rho, \mu, \rho^2)$, $C^x = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, and $C^{\mathbf{h}} = [0 \quad 1 \quad c+1]$. Note that the KOF is globally valid in this case. It is easy to verify that the observability matrix of KOF

$$\mathcal{O} = \begin{pmatrix} C^{\mathbf{h}} \\ C^{\mathbf{h}}A \\ C^{\mathbf{h}}A^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & c+1 \\ 0 & \mu & \rho^2(c+1) \\ 0 & \mu^2 & \rho^4(c+1) \end{pmatrix},$$

is rank deficient. So Theorem I in section (3.3) does not apply. However, null space of $\mathcal{O}$ is linear space spanned by $(1 \quad 0 \quad 0)^*$ and only states $\mathbf{x}$ that gets mapped to this space under $\mathcal{T}_{\mathcal{U}}(\mathbf{x})$ (35) is $\mathbf{x} = 0$. Thus, $\mathbf{x}$ is still observable based on KOF. This example suggests that conditions in Theorem I are not necessary.

Figure 1-a shows that the estimation accuracy of KKF is significantly better that EKF, where the average in (33) is taken over $N_r = 100$ randomly sampled initial conditions around the origin. Note that EKF performs better for the higher noise level, as the noise helps to stabilize the filter. The noisy output and time response of KKF and EKF for one of the sampled initial conditions for $\sigma_o = 1$ is shown in Fig. 1-b.

4.2 Example II

As the second example, we consider discrete time Van der Pol system in reverse time with

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 - x_2 dt \\ x_2 + (x_1 - x_2 + x_1^2 x_2)dt \end{pmatrix}, \tag{36}$$

where, $dt = 0.1$, and let $\mathbf{h}(\mathbf{x}) = x_1^2 + x_2$ be the output. The system has a stable fixed point at origin and an unstable limit cycle which forms the boundary of the basin of attraction for the fixed point, which we will denote by $\mathcal{B}$. To obtain KOF we compute Koopman tuple using kernel EDMD Algo. (2) using snapshot pairs uniformly distributed in $\mathcal{B}$. After trail and error, we found Martern covariance kernel ($K(\mathbf{x}_1, \mathbf{x}_2) = (1 + \frac{\sqrt{5}r}{l} + \frac{5r^2}{3l^2})\exp(-\frac{\sqrt{5}r}{l})$, where, $r = ||\mathbf{x}_1 - \mathbf{x}_2||$, see Rasmussen and Williams (2006)) worked well for this problem.

We used $N = 51$ snapshot pairs leading to computation of $n = 51$ KEs/KMs/KEFs. To numerically study convergence aspects of KOF, we constructed KOF by selecting only a subset $n = 5, 25$ of most dominant KEs/KMs/KEFs based on ranking by norm of the KMs. Note one can also use sparse DMD (Jovanovic et al. (2014)) to select a subset of KMs which are dynamically most relevant. We found that for all cases $n = 5, 25, 51$ the pair $(A, C^{\mathbf{h}})$ for KOF is observable, and thus Theorem I applies. To compare performance of KKF based on KOFs with different order n we used $N_r = 100$ randomly sampled initial conditions in $\mathcal{B}$. The averaged RMSE curves in Fig. 2 show that with increased order of KOF the estimation performance improves as expected, and it is sufficient to retain upto $n = 25$ dominant modes for constructing the KOF.

To compare performance of KKF with EKF we similarly used $N_r = 100$ randomly sampled initial conditions in $\mathcal{B}$. We found that while KKF converged for all the sampled initial conditions, EKF diverged for 21% and 5% initial conditions for the noise levels $\sigma_o = 0.01$ and $\sigma_o = 1$, respectively. The averaged RMSE curves shown in Fig. 3-a are based only on the samples for which EKF converged. The noisy output for $\sigma_o = 0.01$ and time response of KKF and EKF for one of the sampled initial conditions is shown in Fig. 3-b, again superior performance of KKF over EKF is evident.

5. CONCLUSION

In this paper we introduced a new approach for observer synthesis based on Koopman operator theoretic framework. Specifically, we showed how Koopman tuple can be exploited to construct the KOF in a very general setting, thereby making Luenberger/Kalman-like linear observers accessible in a much broader context. Note that while Koopman operator has infinite many eigenvalues/eigenfunctions, only those whose span include the state and output function are required in the KOF. We provided a numerical procedure for constructing such a KOF, and demonstrated superior estimation performance of the KOF based Kalman filter compared to the EKF. By exploiting kernel EDMD approach, the proposed framework can be potentially applied to high dimensional problems.

In future it will be useful to characterize class of nonlinear systems for which an exact finite dimensional KOF exists. For cases where this is not possible, and a truncation is required, further theoretical and numerical studies are required for assessing the tradeoff between level of truncation (i.e. size of KOF) and the estimation performance. It will also be desirable to generalize the KOF based observer synthesis framework for input-output nonlinear systems, a preliminary approach is under development (Surana (Unpublished)).

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