CDS 101/110: Lecture 7.2
Loop Analysis of Feedback Systems

November 7 2016

Goals:

- Review Nyquist Diagrams (including examples).
- *Gain margin* and *phase margin*
- Some analysis of Nyquist stability criterion systems

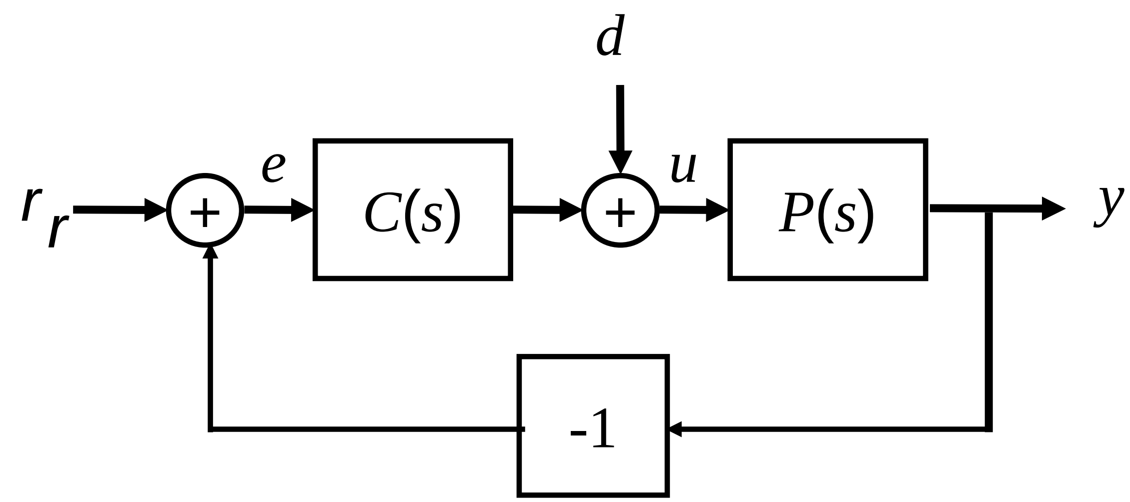
Reading:

- Åström and Murray, Feedback Systems, Chapter 10, Sections 10.1-10.3,

Closed Loop Stability Analysis

"simple" *negative* unity feedback

Q: If we know properties of $P(s)$ and C(s), (the open loop transfer function) can we infer anything about closed loop behavior and performance?



$$L(s) = P(s)C(s) = \frac{n_P(s)}{d_P(s)}\frac{n_C(s)}{d_C(s)}$$

Closed loop transfer function:

$$G_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of G_{yr} determine stability.
- Zeros of $(1 + PC)$ are poles of G_{yr}
- Is there an easy way to check for RHP zeros of $(1 + PC)$?

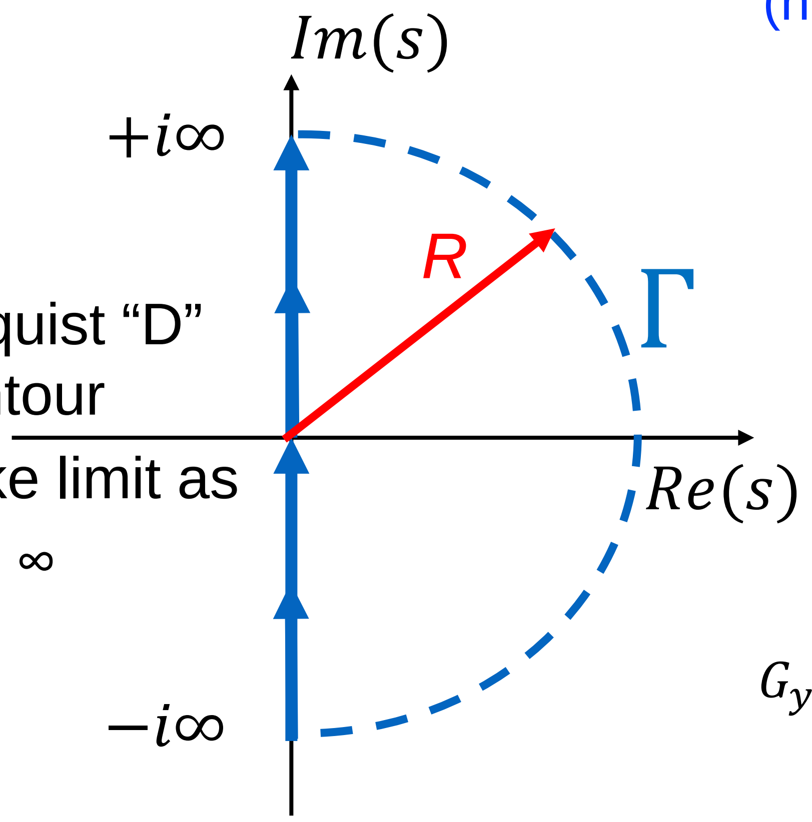
The **Nyquist plot** allows us to check if $(1 + PC)$ has zeros in RHP, and therefore if G_{yr} is unstable

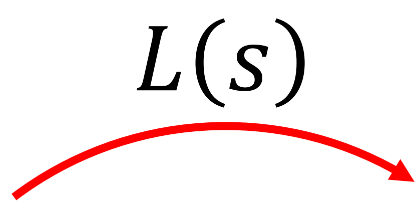
- useful if poles of $d_P(s)d_C(s) + n_P(s)n_C(s)$ not easily found.
- collateral benefit: new ideas on "robustness" of closed loop system.
- useful for ***time delay*** analysis of linear systems

Basic Nyquist Plot (review)

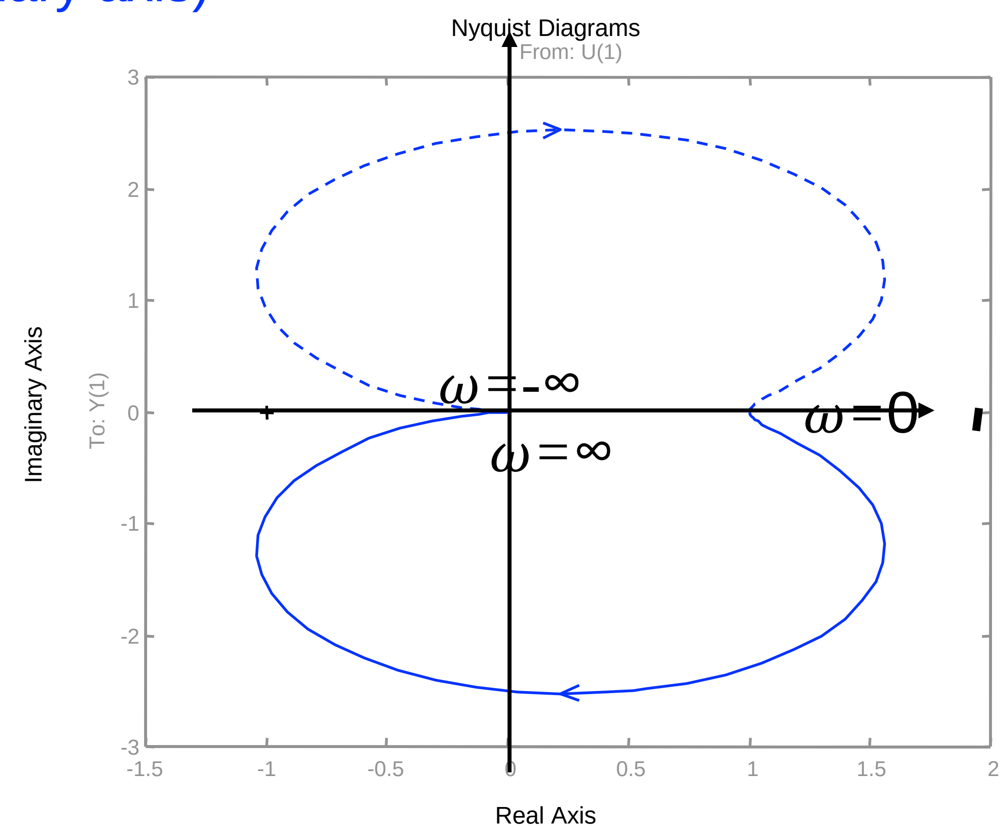
(no poles of $L(s)$ on imaginary axis)

- Nyquist "D" contour
- Take limit as $R \rightarrow \infty$





$$G_{yr}(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



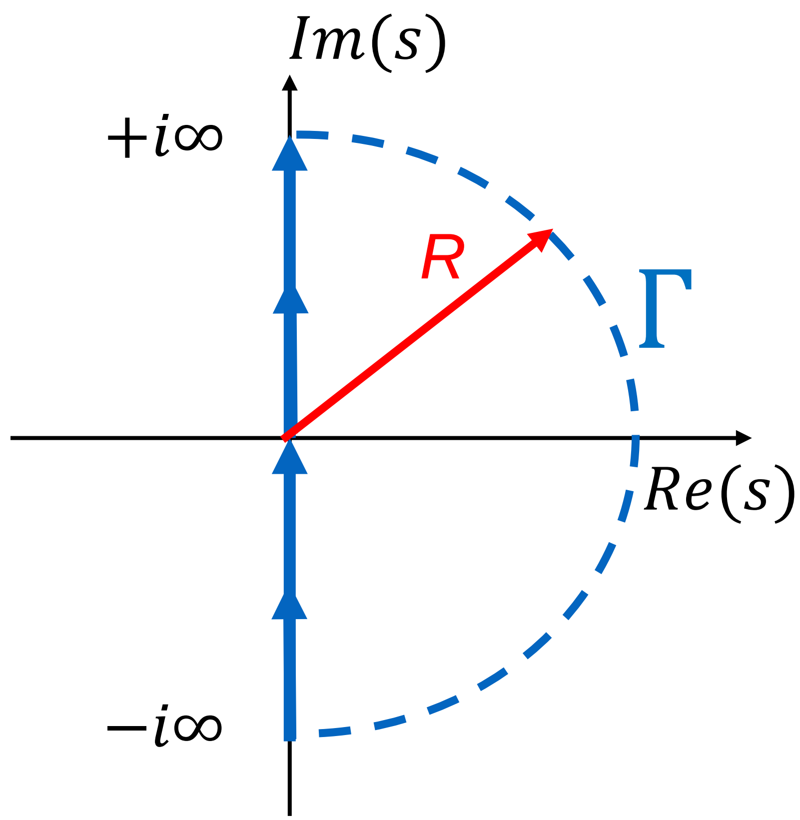
Nyquist Contour (Γ):

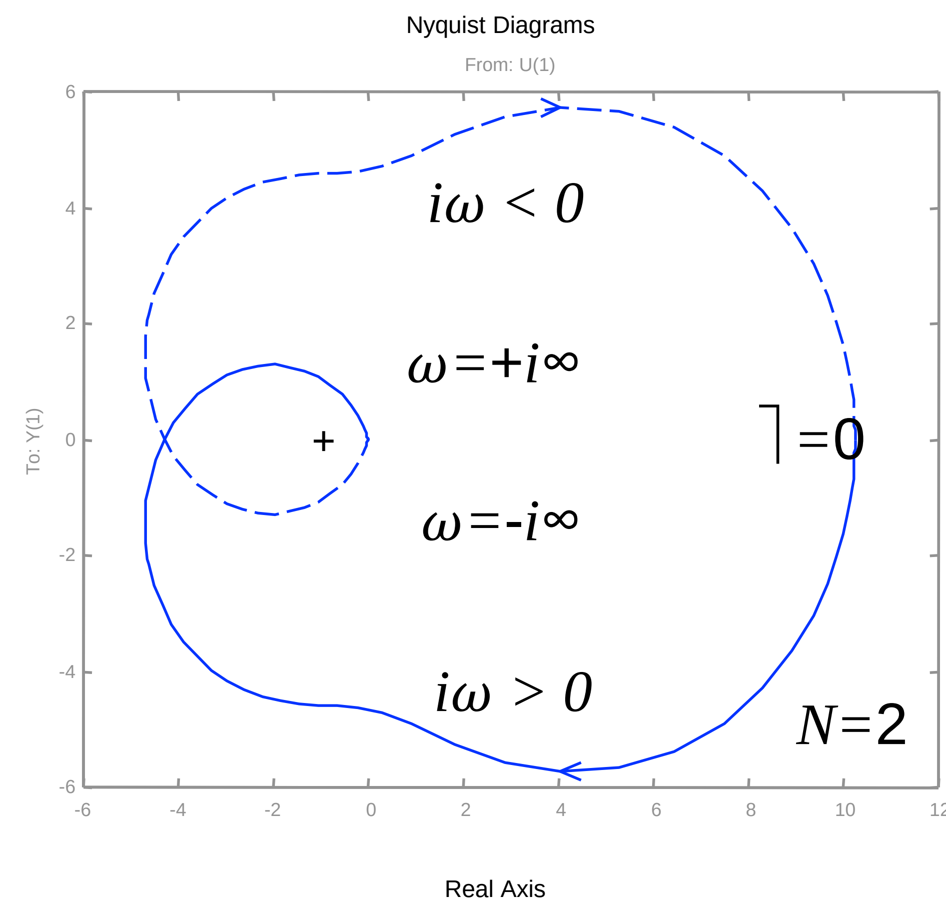
- Start from 0, and move along positive Imaginary axis (increasing frequency)
- Follow semi-Circle, or arc at infinity, in clockwise direction (connecting the endpoints of the imaginary axis)
- From $-i\infty$ to zero on imaginary axis
- Note, portion of plot corresponding to $\omega < 0$ is mirror image of $\omega > 0$

Nyquist Plot

- Formed by tracing s around the Nyquist contour, Γ, and mapping through $L(s)$ to complex plane representing magnitude and phase of $L(s)$.
- I.e., the image of $L(s)$ as s traverses Γ is the Nyquist plot
- **Goal:** from complex analysis, we're trying to find number of zeros (if any) in RHP, which leads to instability

Nyquist Criterion





Thm (Nyquist). Consider the Nyquist plot for loop transfer function *L*(*s*). Let

P # RHP poles of open loop $L(s)$

N # clockwise encirclements of -1 (counterclockwise is negative)

Z # RHP zeros of 1 + $L(s)$

Then

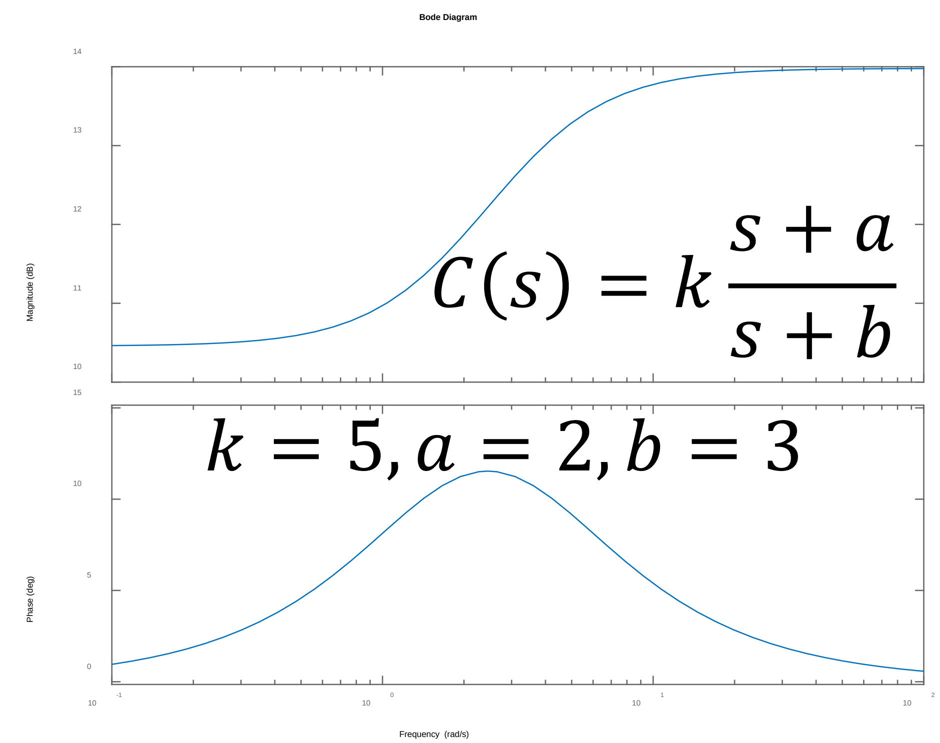
$$Z = N + P$$

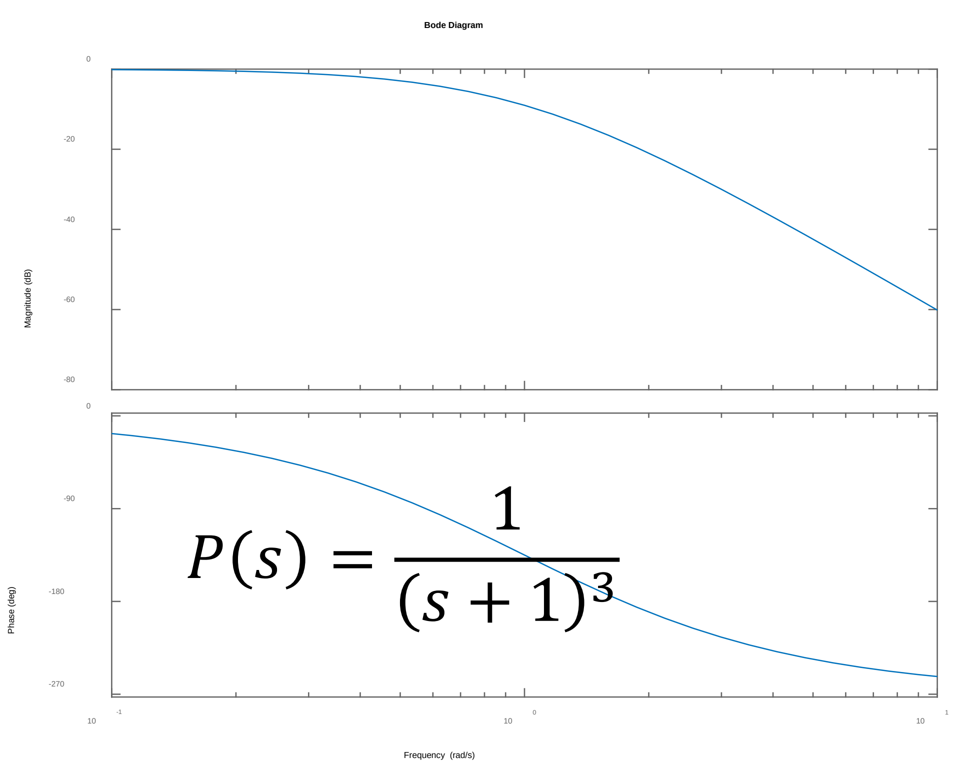
Consequence:

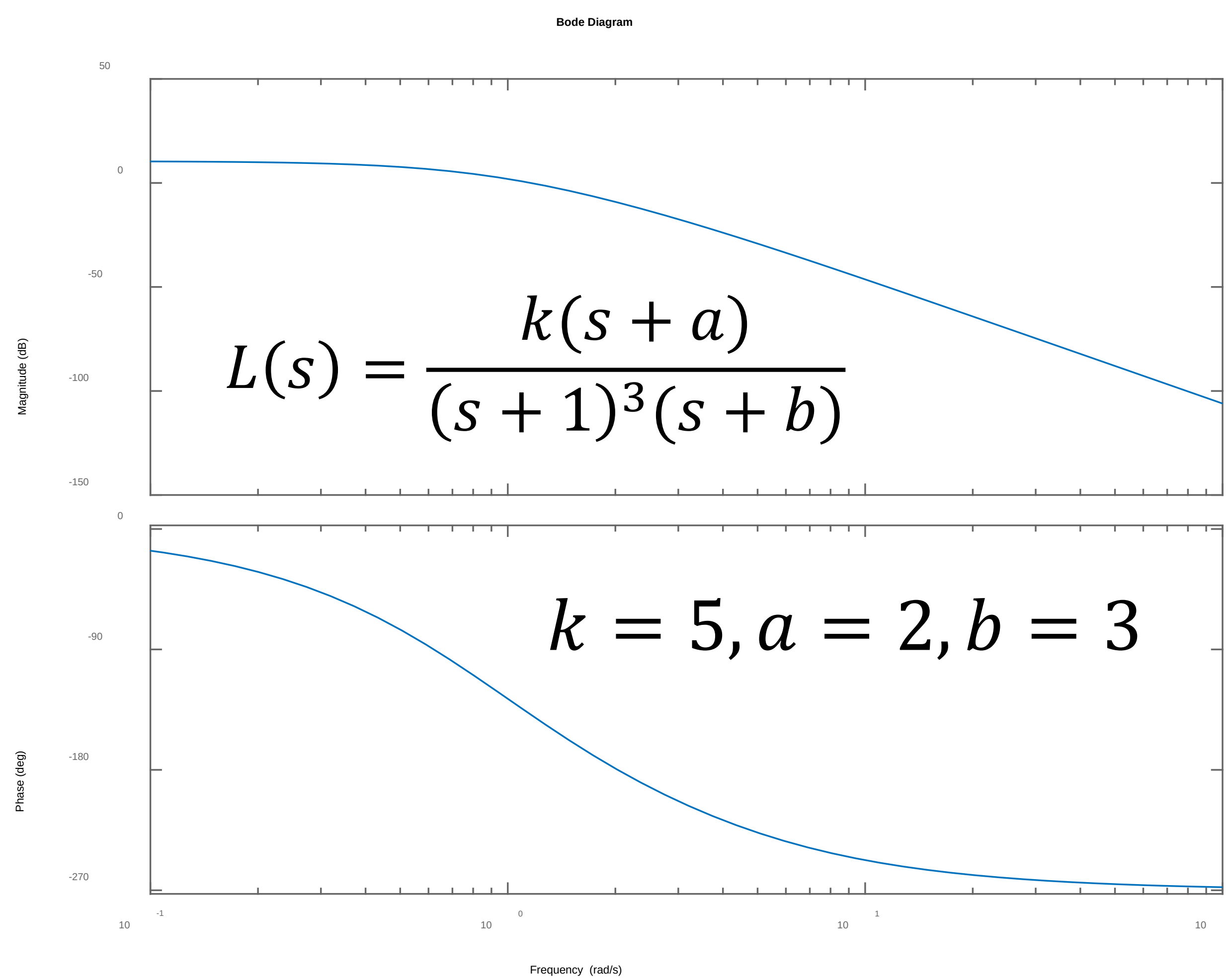
- If $Z \geq 1$, then $(1 + L(s))$ has RHP zeros, which means that $G_{yr}(s)$ has RHP poles.
- $G_{yr}(s)$ is unstable with simple unity feedback, and control $C(s)$

Nyquist Plot Example #1

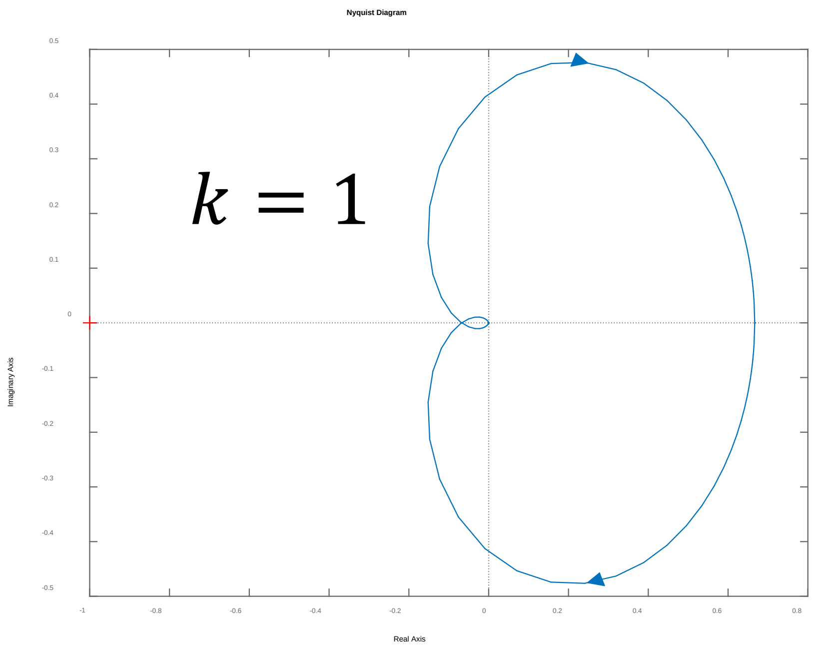
$P(s) = \frac{1}{(s+1)^3}$ $\quad C(s) = k\frac{s+a}{s+b}$ $\quad 1 < a < b$

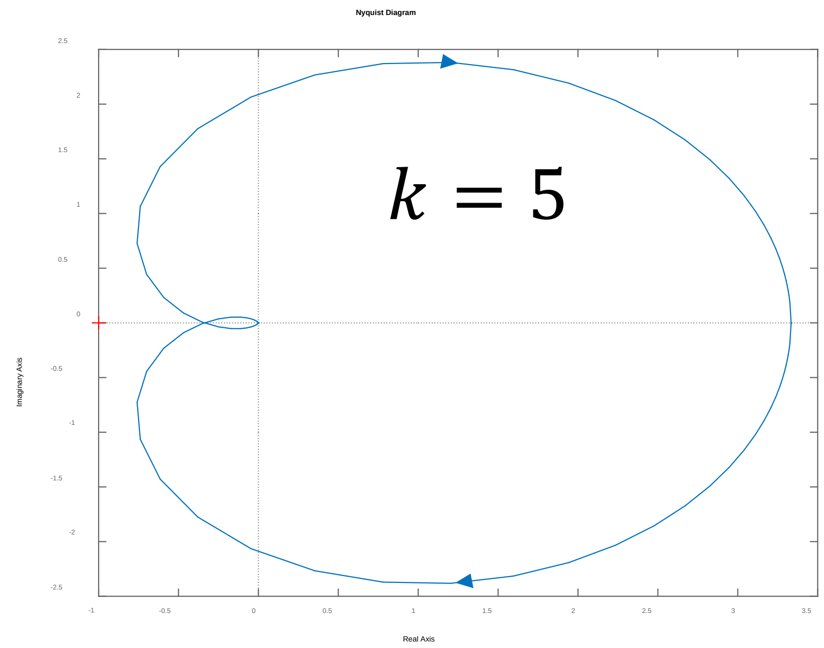


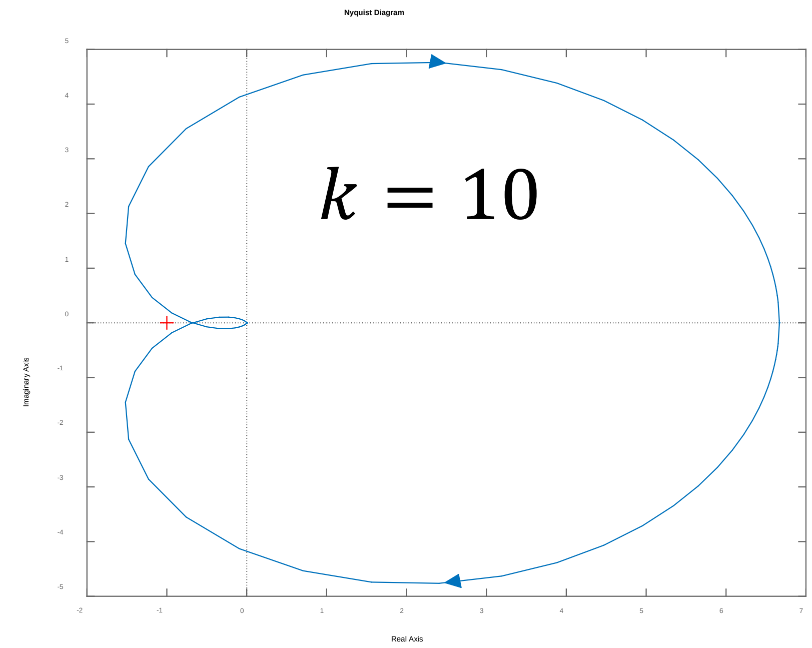




Nyquist Plot Example #1



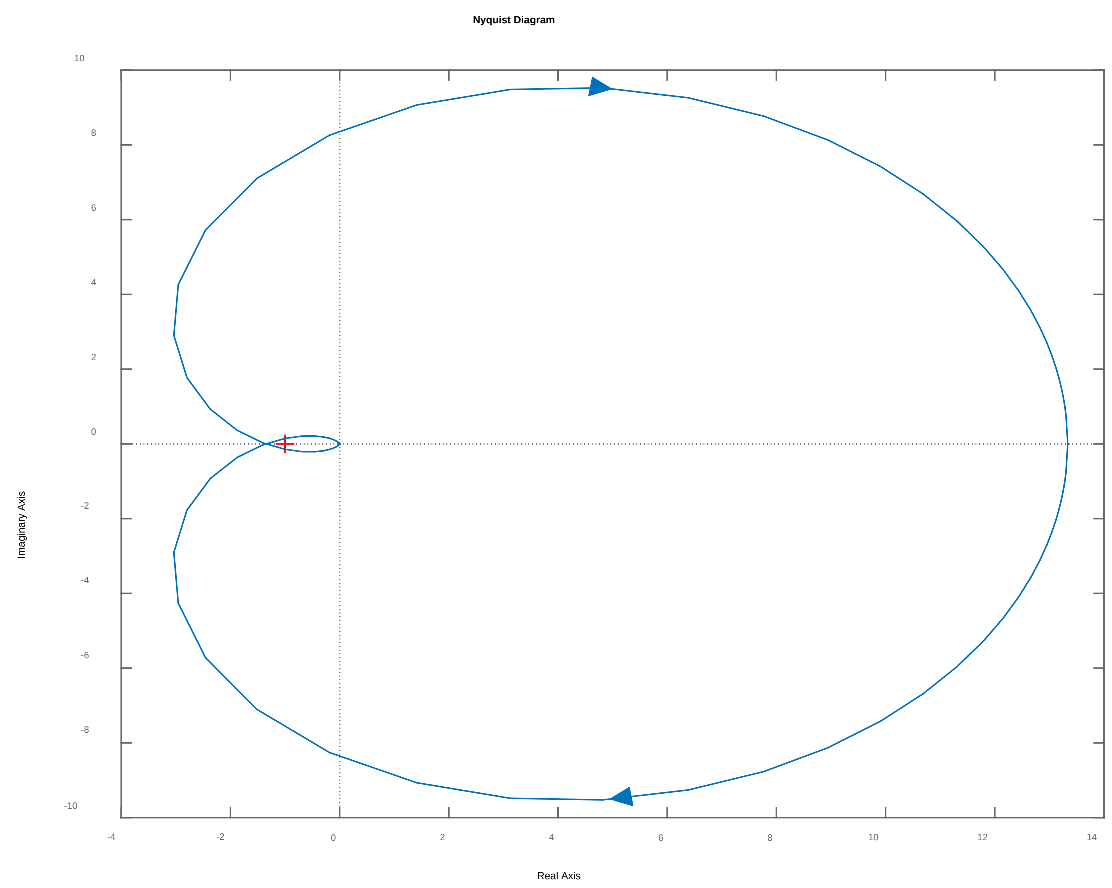




$$P(s) = \frac{1}{(s+1)^3}$$

$$C(s) = k\frac{s+a}{s+b}$$

$$1 < a < b$$



Nyquist:

- $P = 0$
- $N = +1$
- $Z_{RHP} = 1$

Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not how stable

Gain margin

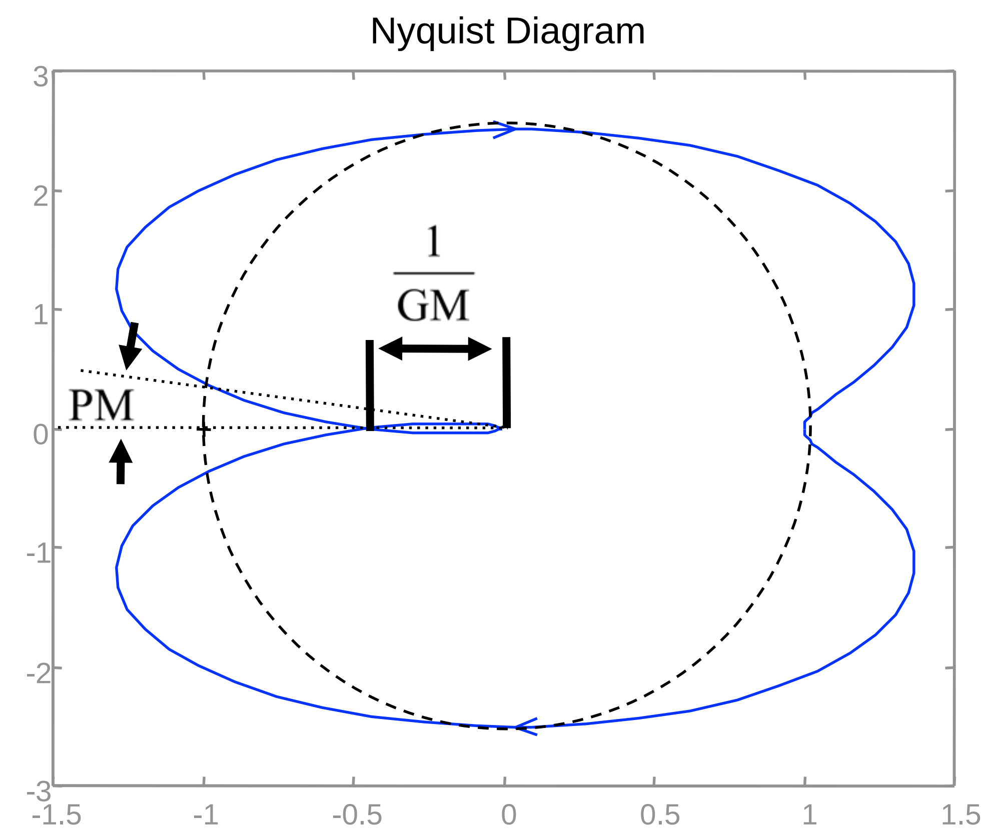
- How much we can modify the loop gain and still have the system be stable
- Determined by the location where the loop transfer function crosses 180˚ phase

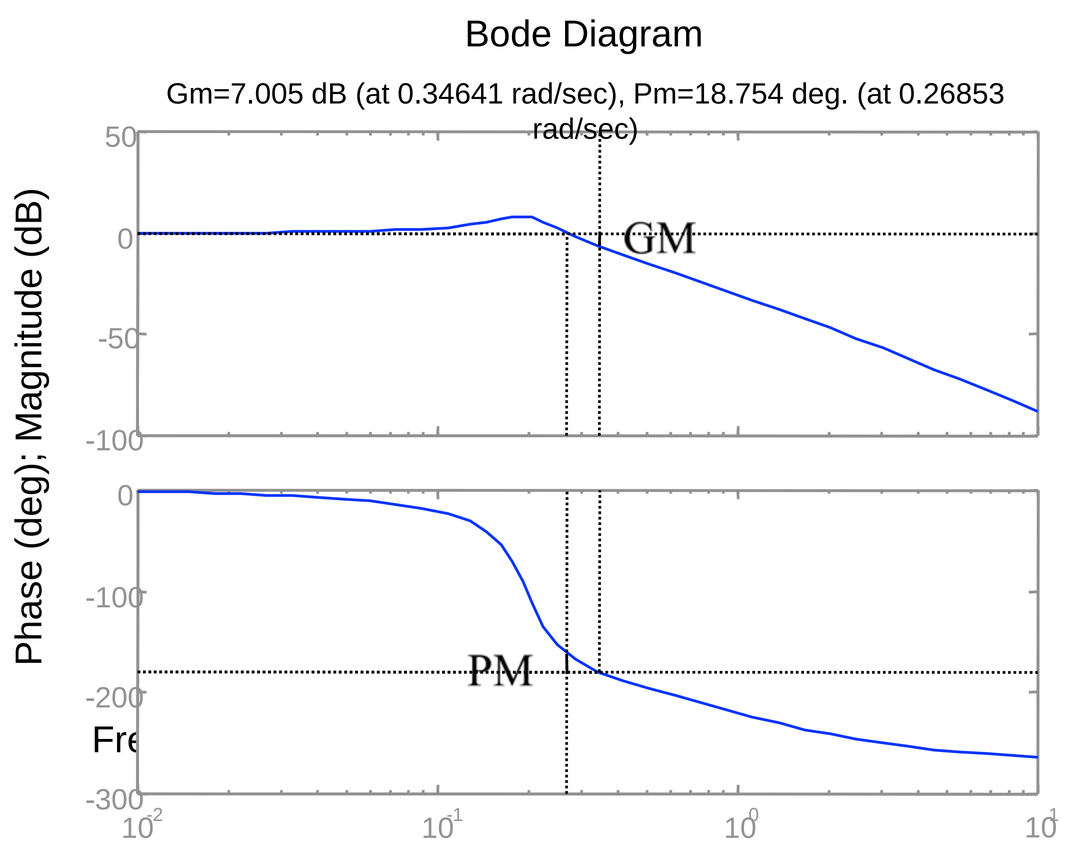
Phase margin

- How much "phase delay" can be addeded while system remains stable
- Determined by the phase at which the loop transfer function has unity gain

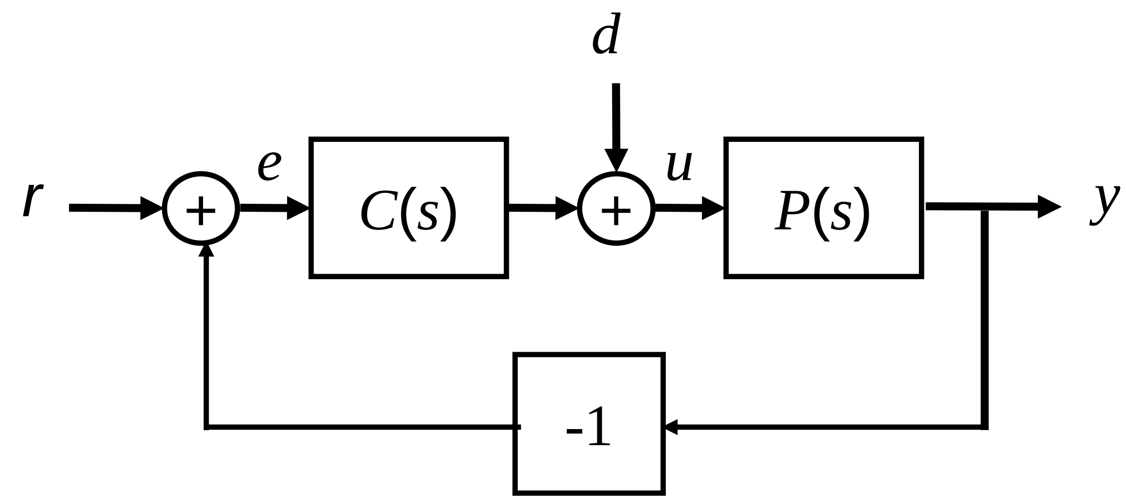
Bode plot interpretation

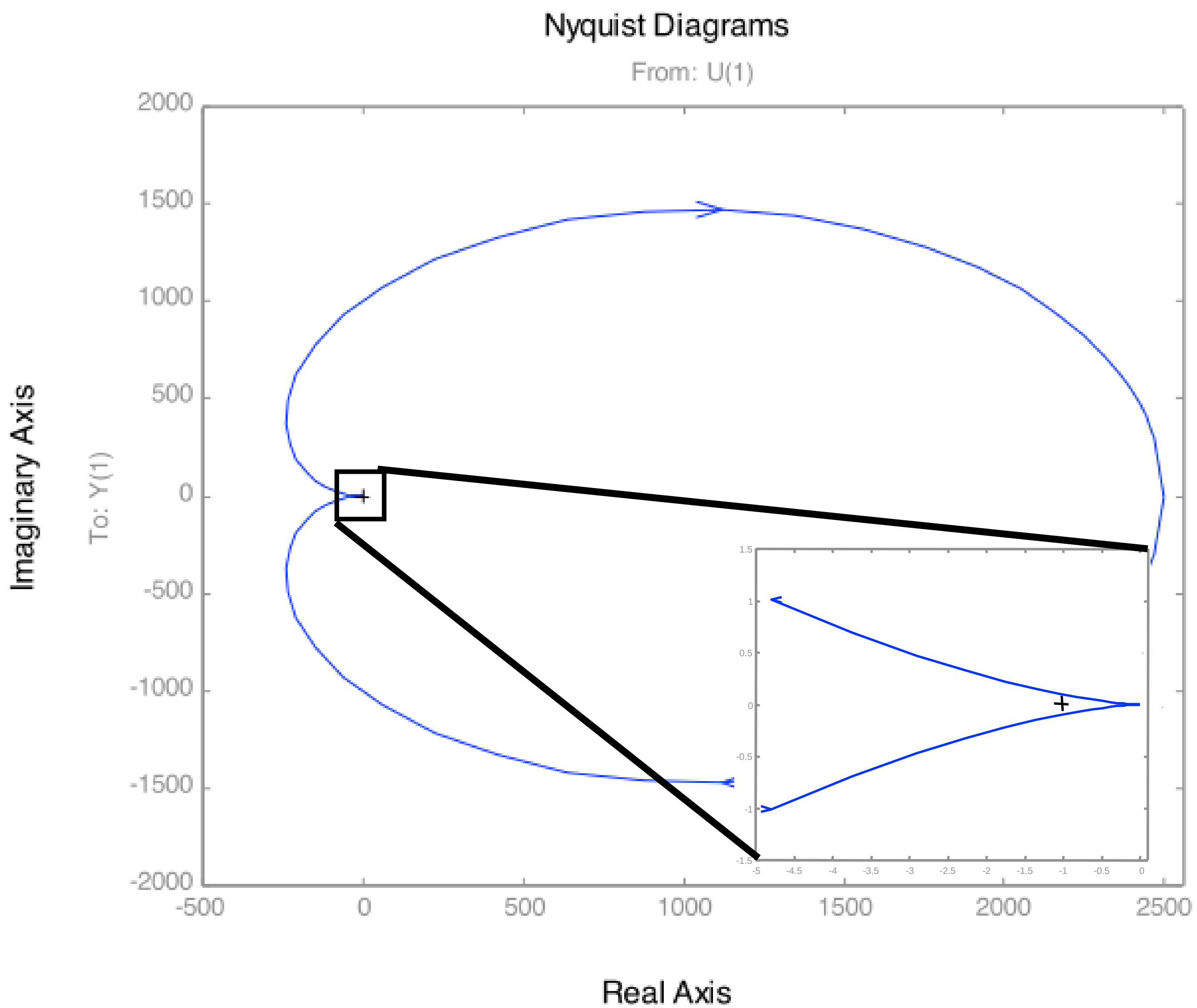
- Look for gain = 1, 180˚ phase crossings
- MATLAB: margin(sys)





Example: Proportional + Integral* speed controller





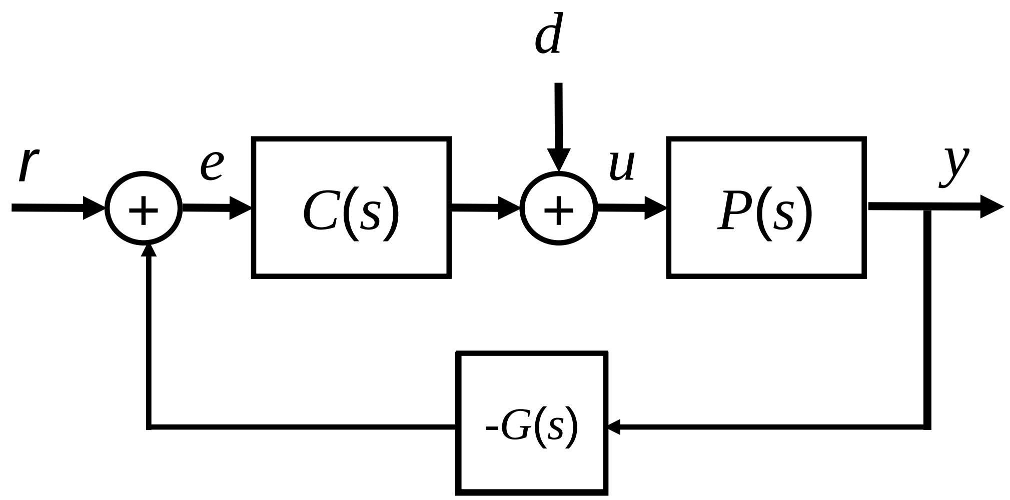
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

Remarks

- N = 0, P = 0 ⇒ Z = 0 (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don't have to compute closed loop response

Example: cruise control



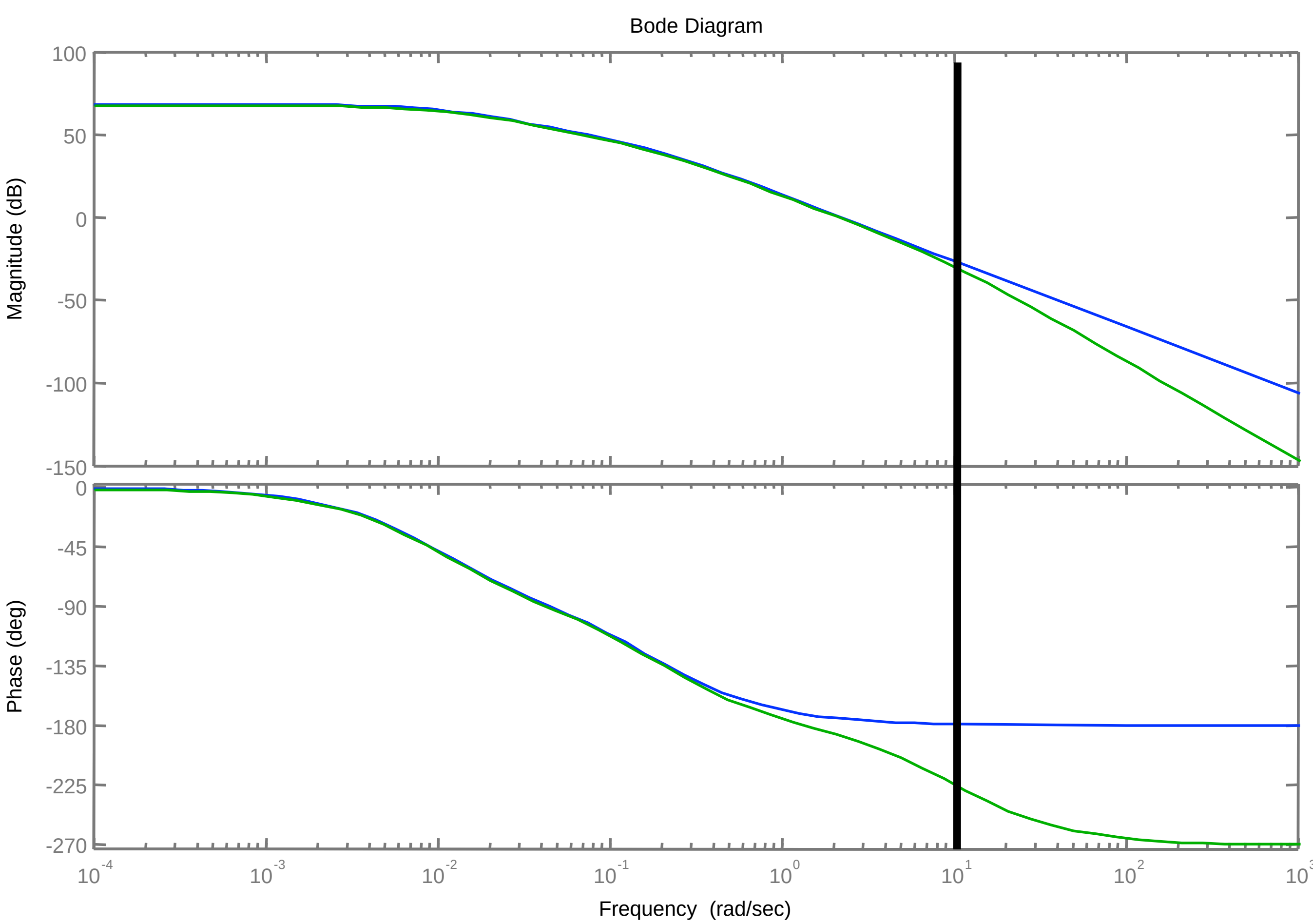
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

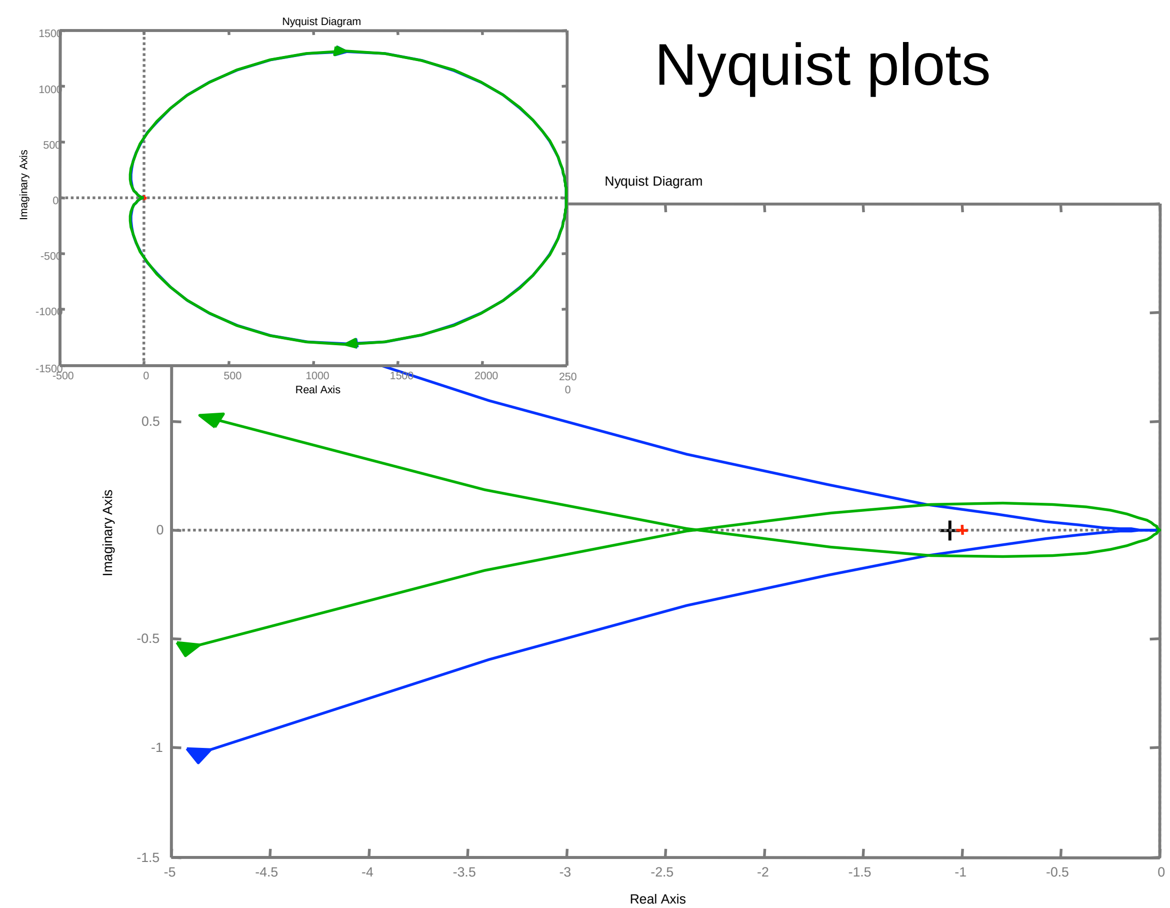
$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s + 10}$$

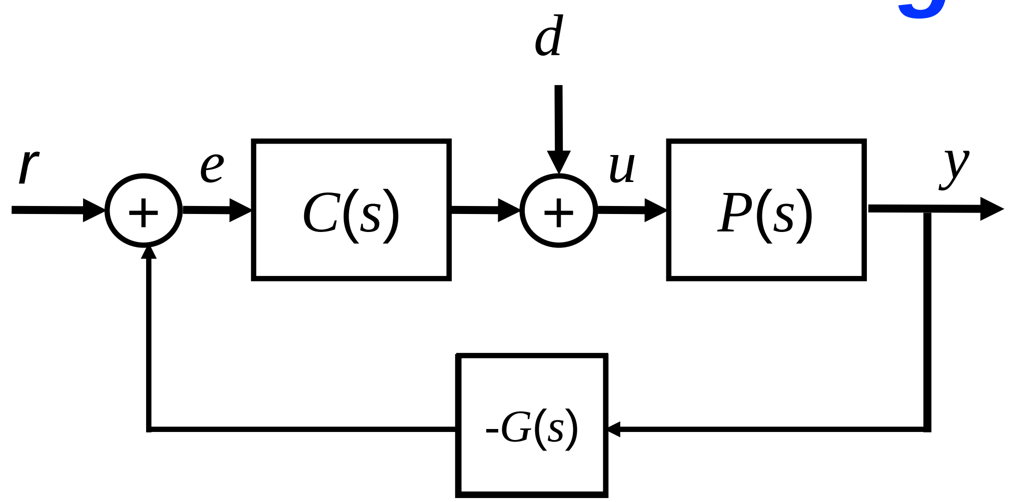
Effect of additional sensor dynamics

- New speedometer has pole at s = 10 (very fast); problems develop in the field
- What's the problem? A: insufficient phase margin in original design (not robust)





Preview: control *design*



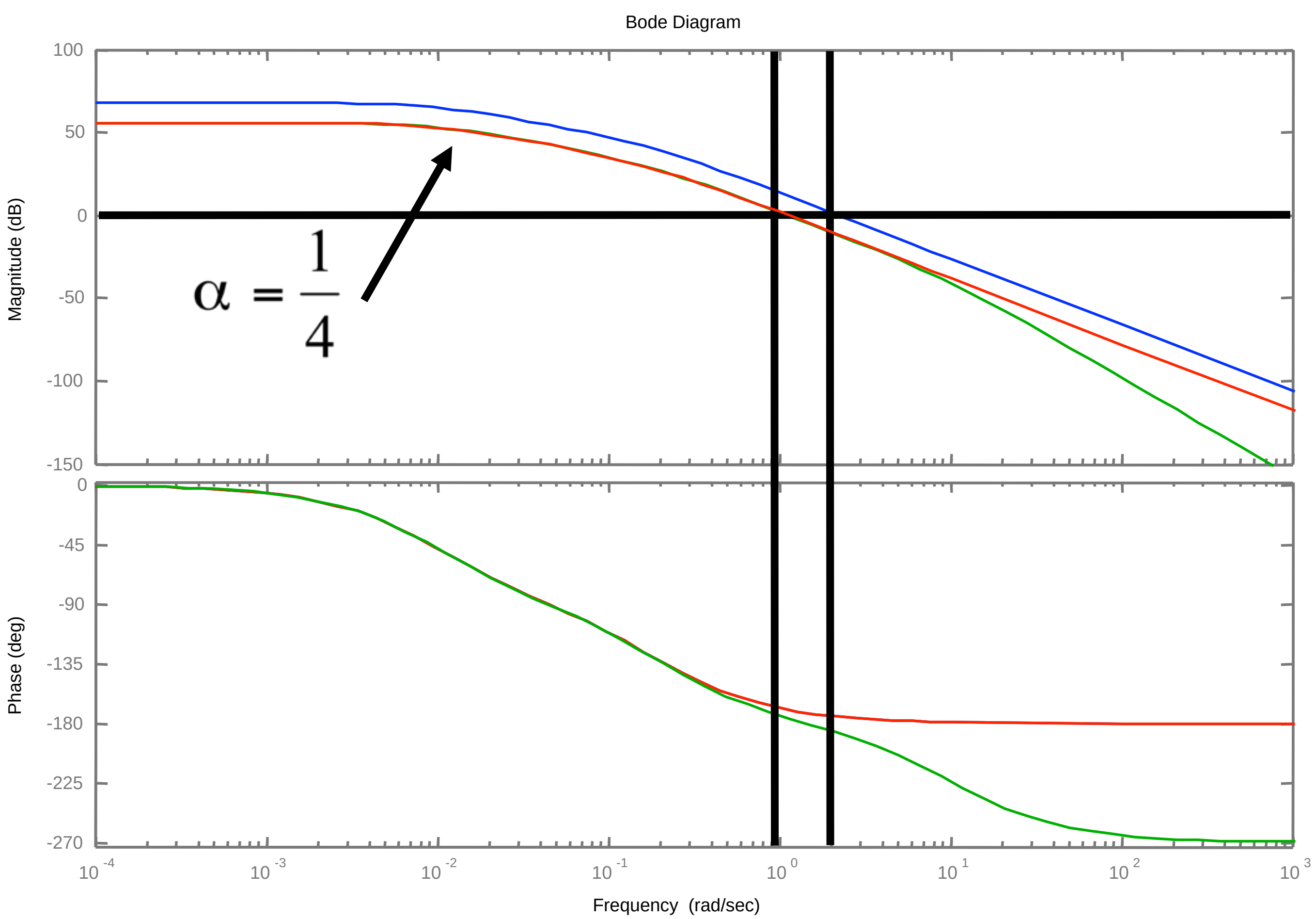
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

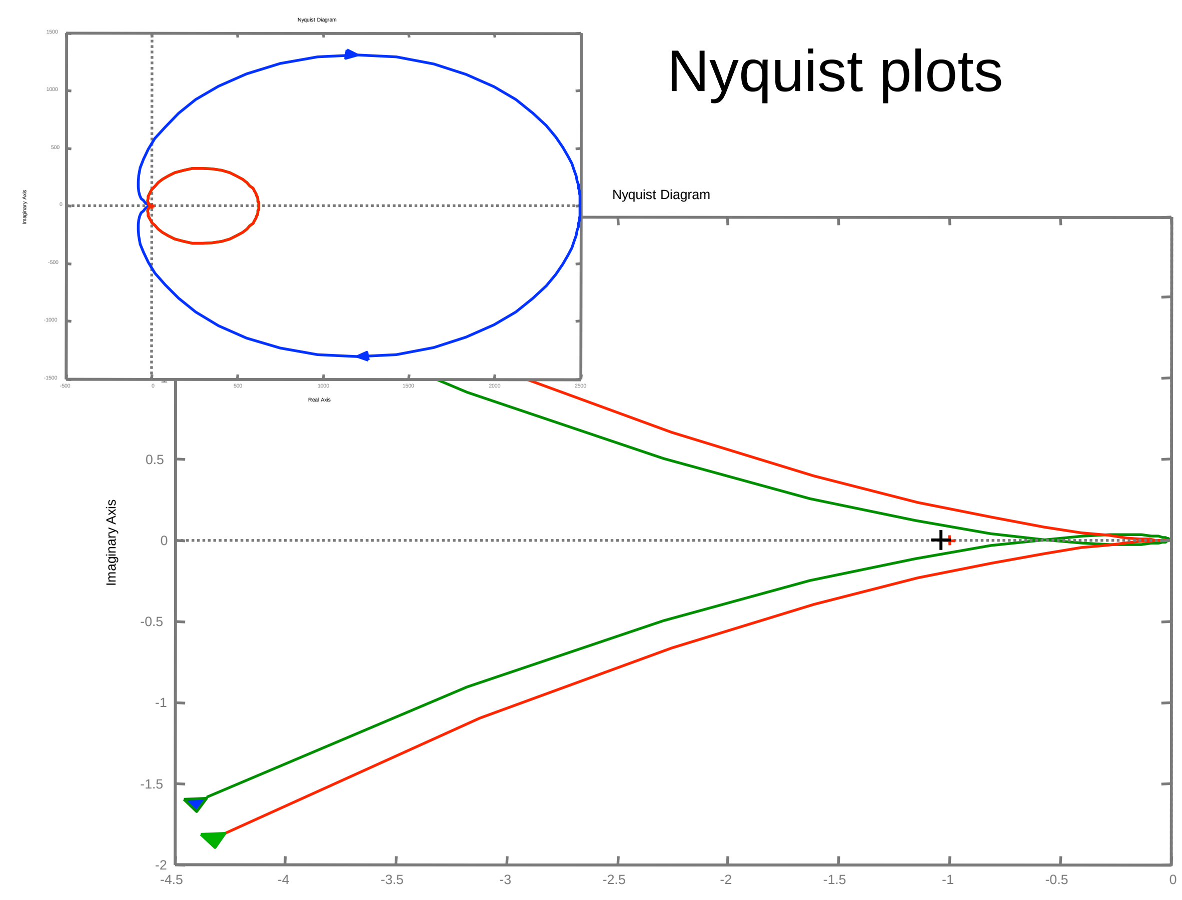
$$C(s) = \alpha \left(K_p + \frac{K_i}{s + 0.01} \right)$$

$$G(s) = \frac{10}{s+10}$$

Approach: Increase phase margin

- Increase phase margin by reducing gain ⇒ can accommodate new sensor dynamics
- Tradeoff: lower gain at low frequencies ⇒ less bandwidth, larger steady state error



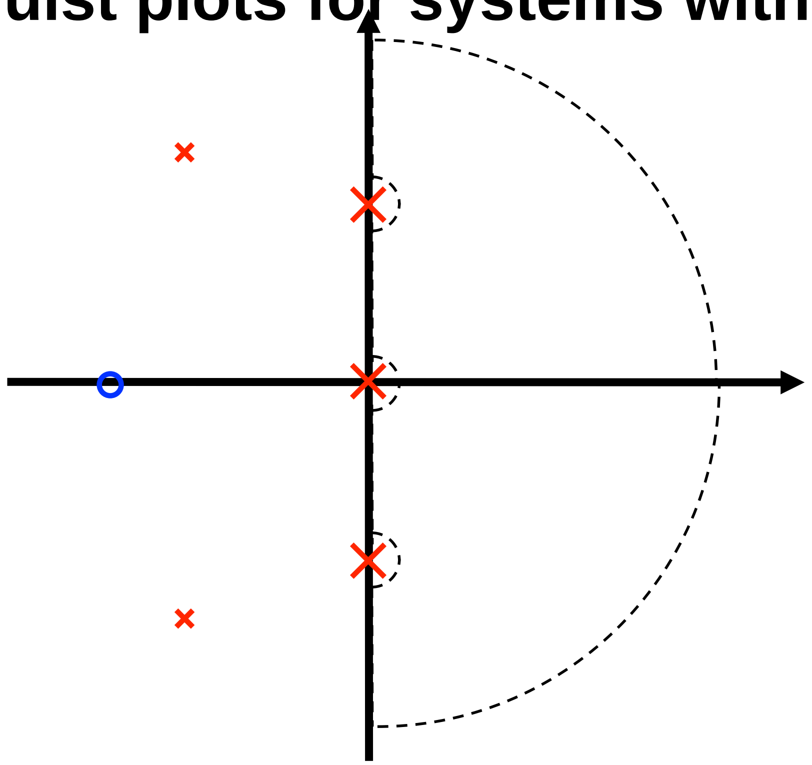


Comments and cautions

Why is the Nyquist plot useful?

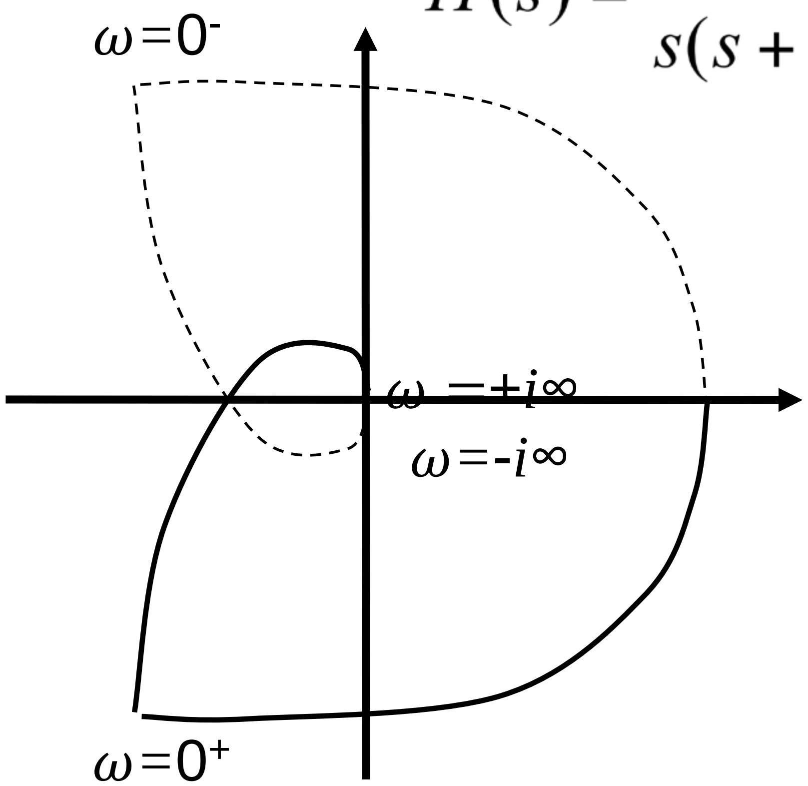
- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives insight into stability and robustness; very useful for reasoning about stability

Nyquist plots for systems with poles on the jω axis



- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\varepsilon+0i)$ to determine direction

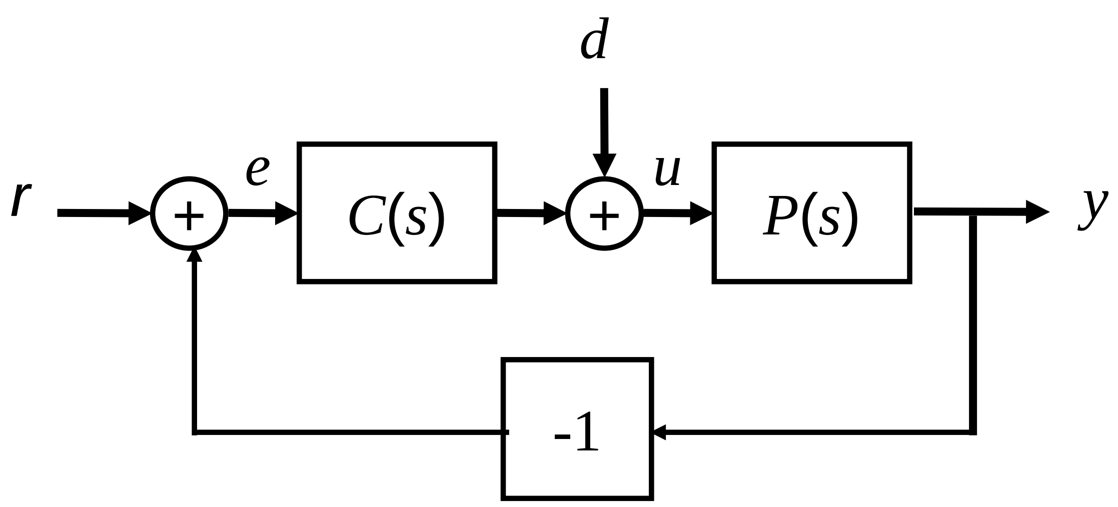
$$H(s) = \frac{1}{s(s+1)^2}$$

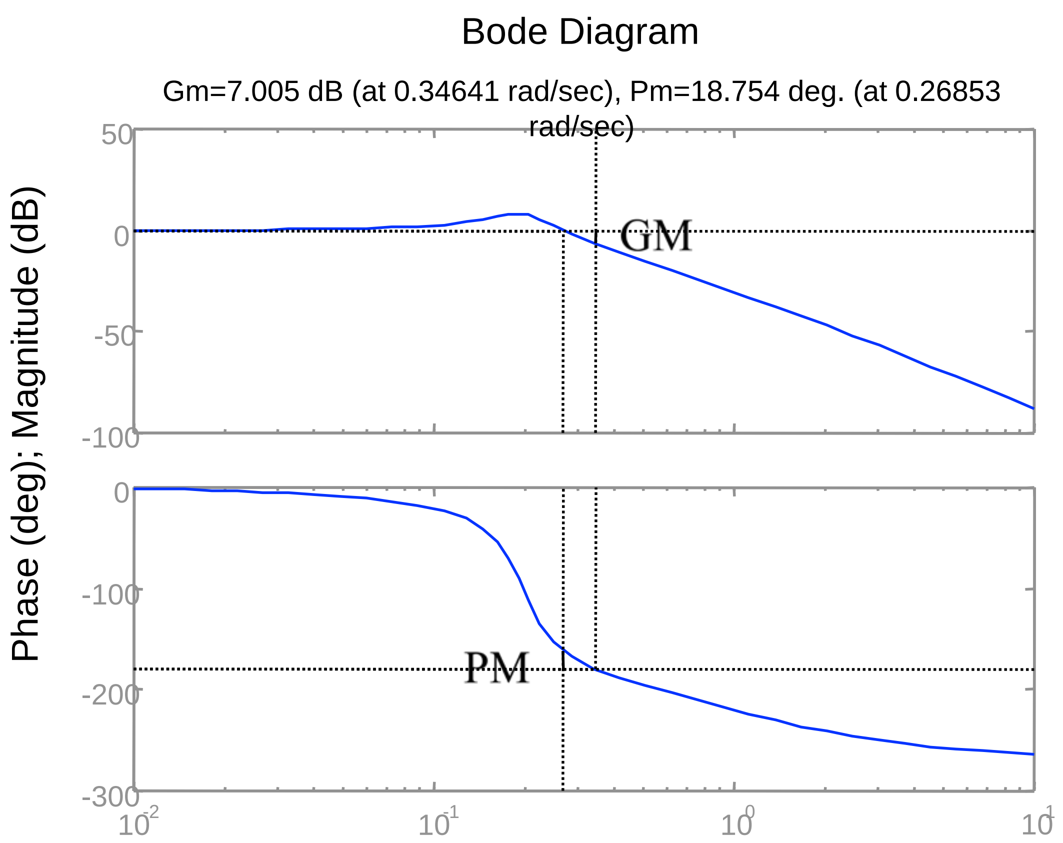


Cautions with using MATLAB

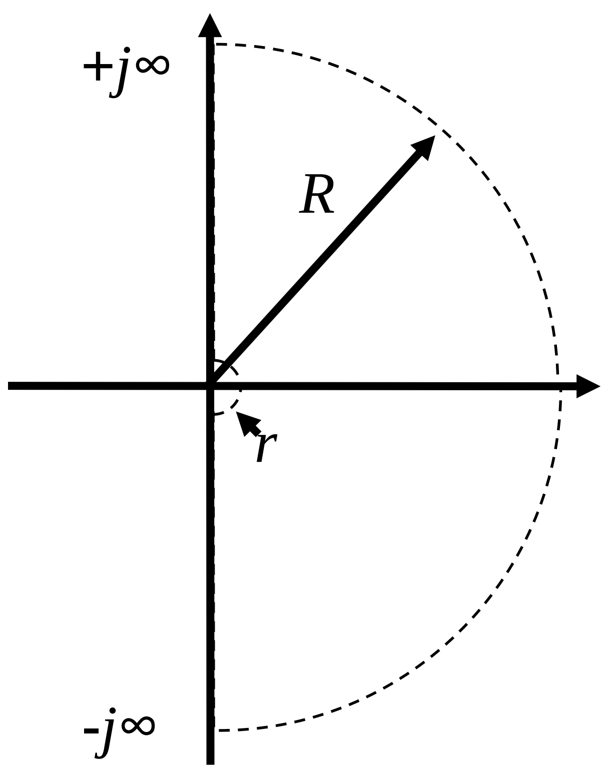
- MATLAB doesn't generate portion of plot for poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)

Summary: Loop Analysis of Feedback Systems





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



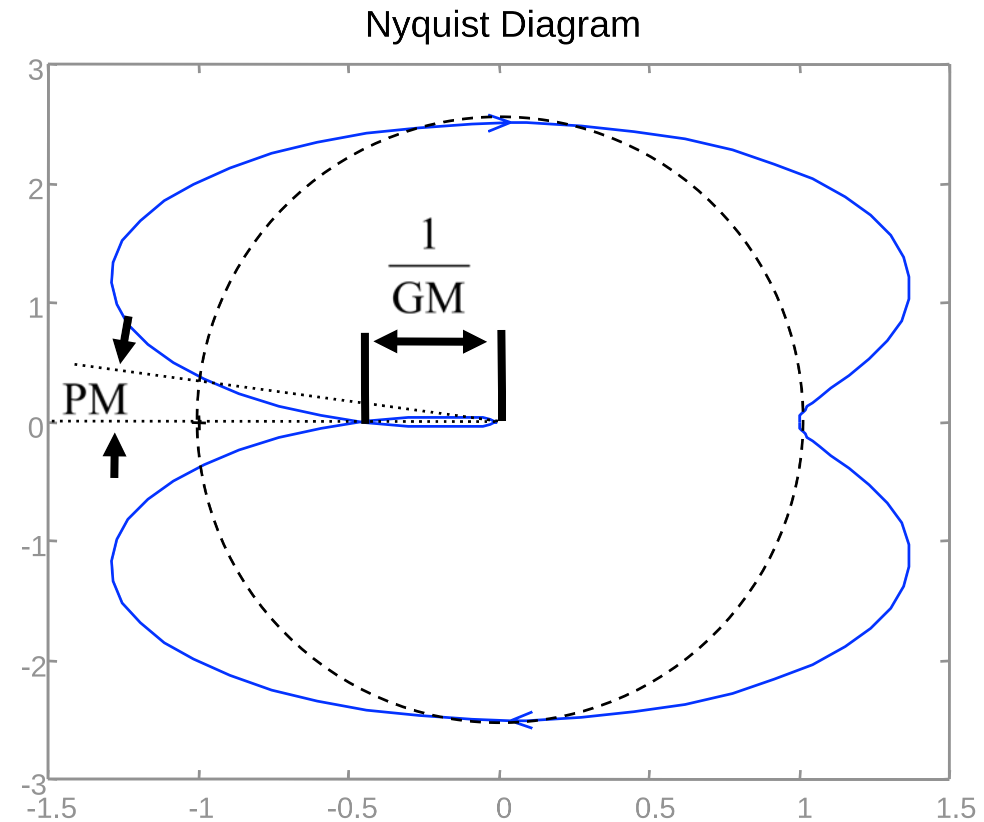
Thm (Nyquist).

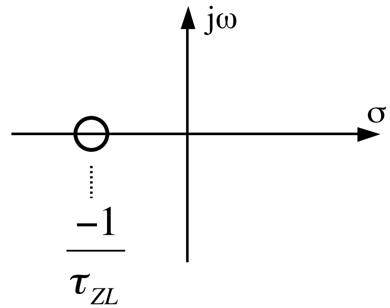
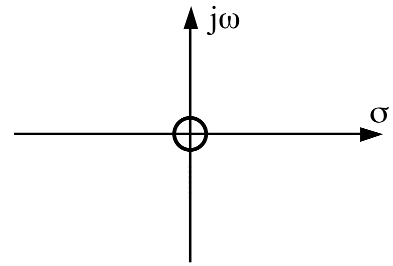
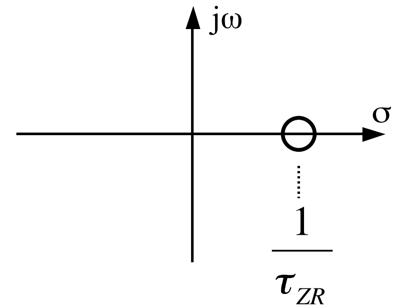
P # RHP poles of $L(s)$

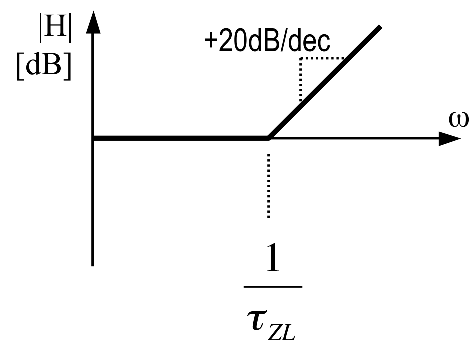
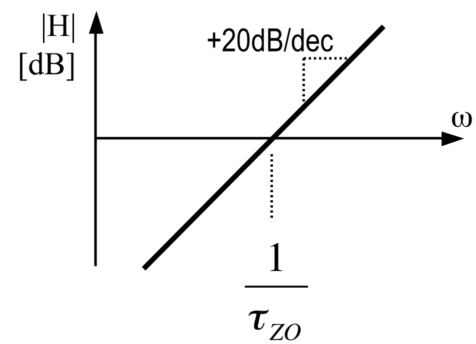
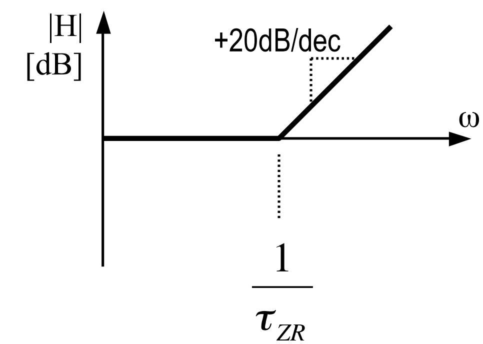
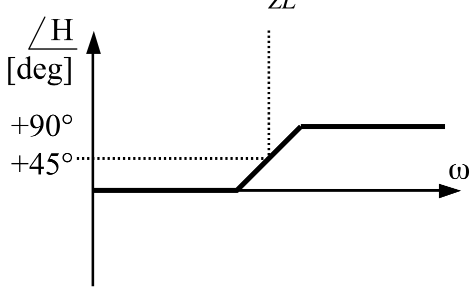
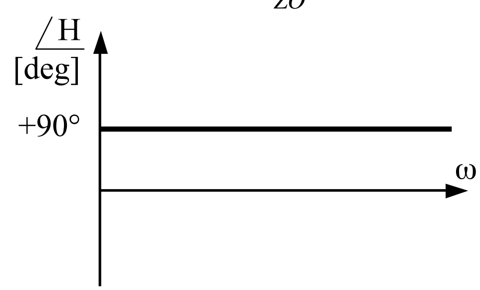
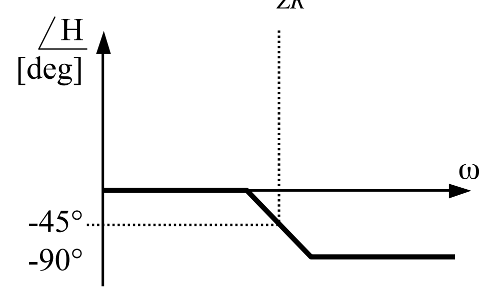
N # CW encirclements

Z # RHP zeros

$$Z = N + P$$



<table>
<tr><th></th><th>LHP ZERO</th><th>ZERO AT ORIGIN (DERIVATIVE)</th><th>RHP ZERO</th></tr>
<tr><td>TERM IN H(s)</td><td>$1 + s\tau_{ZL}$</td><td>$s\tau_{ZO}$</td><td>$1 - s\tau_{ZR}$</td></tr>
<tr><td>S-PLANE</td><td></td><td></td><td></td></tr>
<tr><td>BODE PLOT:
LET s = jω

MAGNITUDE</td><td>$\lvert 1 + j\omega\tau_{ZL} \rvert = \sqrt{1^2 + (\omega\tau_{ZL})^2}$
</td><td>$\lvert 0 + j\omega\tau_{ZO} \rvert = \omega\tau_{ZO}$
</td><td>$\lvert 1 - j\omega\tau_{ZR} \rvert = \sqrt{1^2 + (\omega\tau_{ZR})^2}$
</td></tr>
<tr><td>PHASE</td><td>$\angle 1 + j\omega\tau_{ZL} = \tan^{-1}(\omega\tau_{ZL})$</td><td>$\angle 0 + j\omega\tau_{ZL} = \tan^{-1}\left(\frac{\omega\tau_{ZL}}{0}\right) = +90^\circ$</td><td>$\angle 1 - j\omega\tau_{ZR} = \tan^{-1}(-\omega\tau_{ZR})$</td></tr>
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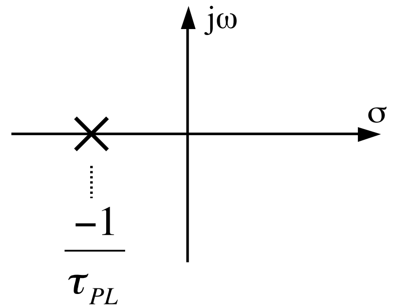
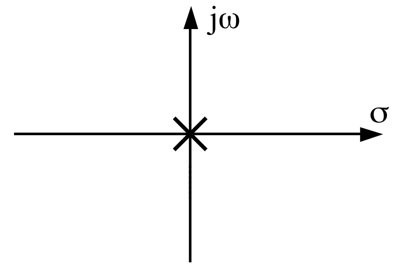
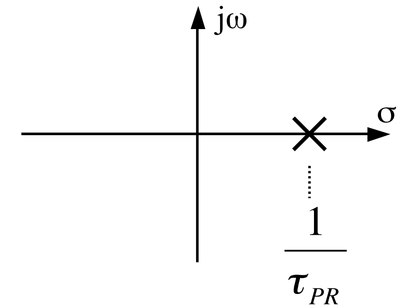
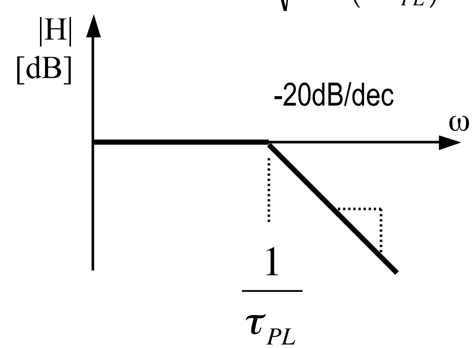
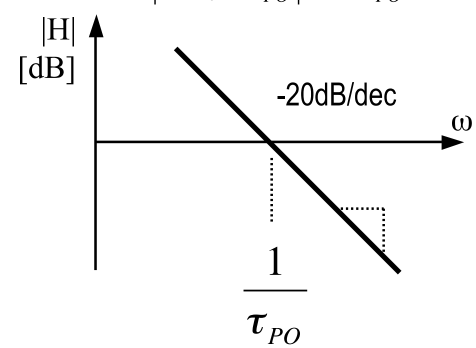
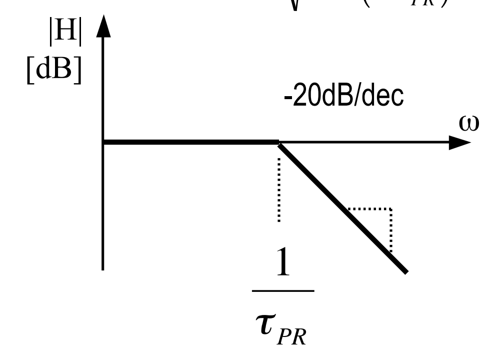
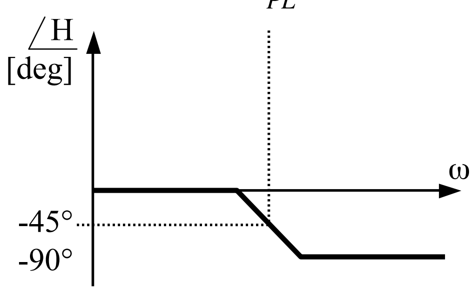
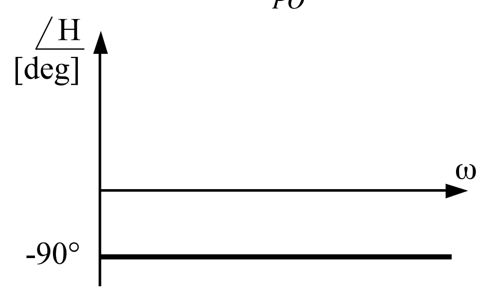
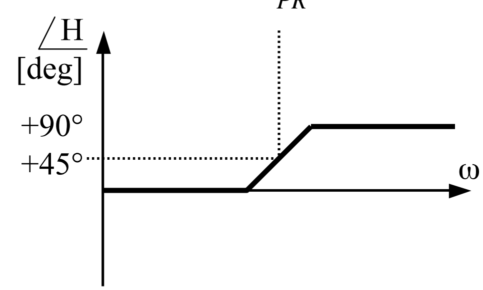
	LHP POLE	POLE AT ORIGIN (INTEGRATOR)	RHP POLE
TERM IN H(s)	$\dfrac{1}{1+s\tau_{PL}}$	$\dfrac{1}{s\tau_{PO}}$	$\dfrac{1}{1-s\tau_{PR}}$
S-PLANE			
BODE PLOT: LET s = jω MAGNITUDE	$\left\lvert\dfrac{1}{1+j\omega\tau_{PL}}\right\rvert=\dfrac{1}{\sqrt{1^2+(\omega\tau_{PL})^2}}$ 	$\left\lvert\dfrac{1}{0+j\omega\tau_{PO}}\right\rvert=\dfrac{1}{\omega\tau_{PO}}$ 	$\left\lvert\dfrac{1}{1-j\omega\tau_{PR}}\right\rvert=\dfrac{1}{\sqrt{1^2+(\omega\tau_{PR})^2}}$ 
PHASE	 $\angle 1/(1+j\omega\tau_{PL})=-\tan^{-1}(\omega\tau_{PL})$	 $\angle 1/(0+j\omega\tau_{PO})=-\tan^{-1}(\omega\tau_{PO}/0)=-90^\circ$	 $\angle 1/(1-j\omega\tau_{PR})=\tan^{-1}(\omega\tau_{PR})$