CDS 101/110: Lecture 5.2
Observability & State Estimation

October 28, 2016

Goals:

- Review Observability and Observers.
- Complete and “polish” the analysis of combined feedback and observation.
- A few thoughts on observer design.
- Brief mid-term review

Reading:

- Åström and Murray, Feedback Systems-2e, Section 8.1-8.3

Observability

System: $\dot{x} = Ax + Bu; \quad y = Cx + Du$ (*)

- **Definition:** The linear system (*) is said to be ***Observable*** if for every T>0 it is possible to determine the system state $x(T)$ through measurements $y(t)$ and knowledge of $u(t)$ on the interval $[0, T]$.
  - Note: some texts/papers are slightly different: Observable if $x(t = 0)$ can be determined from measurements and inputs.
  - If (*) is observable, then there are no "hidden" internal states. This is a practical issue in system design—do you have the right sensors?

Testing for Observability:

- The Matrix, W_O must be full rank $\quad W_O \equiv \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Observable Canonical Form

System: $\dot{x} = Ax + Bu; \quad y = Cx + Du \qquad (*)$

- **Definition:** The linear system (*) is said to be in ***Observable Canonical Form* (OCF)** if

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0 \quad \cdots \quad 0]x + d_0 u$$

Where the characteristic polynomial of A is: $\lambda_A(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0$

- When the system (*) is in OCF, the controllability matrix takes the form:

$$\widetilde{W}_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_1 & 1 & 0 & \cdots & 0 \\ -a_1^2 - a_2 & -a_1 & \ddots & \ddots & 0 \\ \vdots & \vdots & * & \cdots & \vdots \\ * & * & * & \cdots & 1 \end{bmatrix}; \quad \widetilde{W}_O^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_1 & 1 & 0 & \cdots & 0 \\ -a_1^2 - a_2 & -a_1 & \ddots & \ddots & 0 \\ \vdots & \vdots & * & \cdots & \vdots \\ * & * & * & \cdots & 1 \end{bmatrix}$$

State Estimation

State Estimator: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

- The term $L(y - C\hat{x})$ provides "feedback" to the estimation process.
- **Analysis:** Let $\tilde{x} = x - \hat{x}$ denote the *error* in the state estimate. Then

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - [A\hat{x} + Bu + L(y - C\hat{x})]$$
$$= A(x - \hat{x}) + LC(x - \hat{x}) = (A - LC)\tilde{x}$$

Hence, the converge of the estimation error is governed by eigenvalues of $(A - LC)$

- Dual to previous reachability analysis. "Design" = eigenvalues of $(A - LC)$.
- Place poles of $(A^T - C^T L^T)$. MATLAB: *place*$(A^T, C^T, eigenvalues)$

- **Theorem:** If (A, C) is *observable,* then the poles of $(A - LC)$ can be set arbitrarily.
- **Design:** Specify the desired poles of $(A - LC)$ by

$$\lambda_{A-LC}(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n = 0$$

Then gain matrix is found as: $L = W_O^{-1}\widetilde{W}_O \begin{bmatrix} p_1 - a_1 \\ \vdots \\ p_n - a_n \end{bmatrix}$

Feedback of Estimated State

Feedback the estimated state: $u = -K\hat{x} + k_r r$

- **Analysis:** Again, let $\tilde{x} = x - \hat{x}$ denote the error in the state estimate. The dynamics of the controlled system under this feedback are:

$$\dot{x} = Ax + Bu = Ax - BK\hat{x} - Bk_r r = Ax - BK(x - \tilde{x}) + Bk_r r$$
$$= (A - BK)x + BK\tilde{x} + Bk_r r$$

  - Introduce a new *augmented* state: $q = [x \quad \tilde{x}]^T$. The dynamics of the system defined by this state is:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} (A - BK) & BK \\ 0 & (A - LC) \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r \equiv Mq + B_M r$$

  The characteristic polynomial of M is:

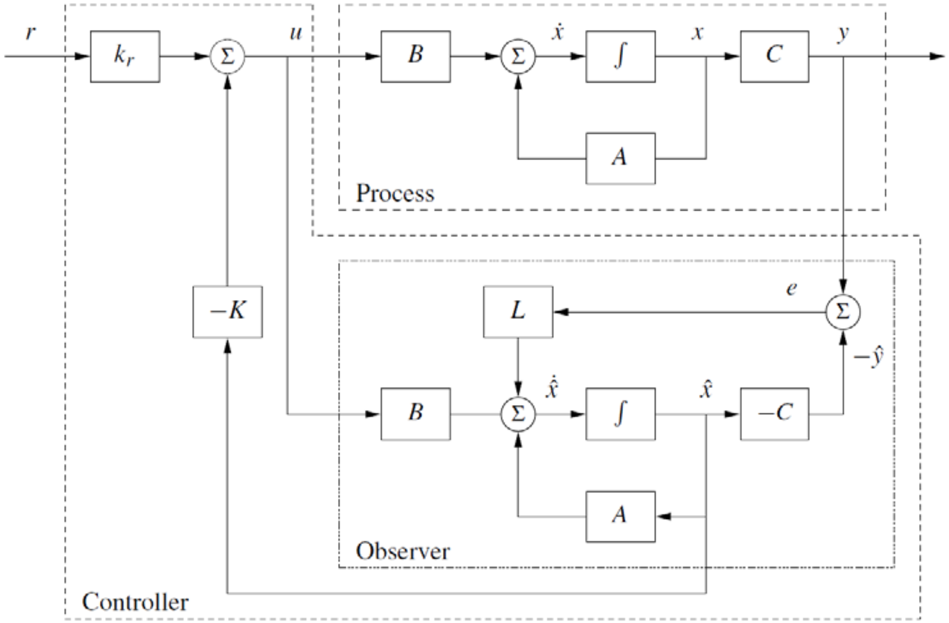
$$\lambda_M(s) = \det(sI - A + BK)\det(sI - A + LC)$$

- If the system is *observable* and *reachable*, then the poles of $(A - BK)$ and $(A - LC)$ can be set *arbitrarily* and *independently*

Feedback of Estimated State

Remarks:

- The controller is a dynamical system with internal state dynamics (the observer).
- *Separation principle:* The controller and observer can be designed (eigenvalues assigned) separately/independently.
- *Internal Model principle:* the control system includes and *internal model* of the system being controlled.



Reachability

For LTI system $\dot{x} = Ax + Bu, \quad y = Cx + Du,$ reachability assessed by rank of:

$$W_r = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

Definitions: *recall* $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

- ***Controllable*** if state can be driven to $x(T) = 0$ for any $x(0)$
  - i.e., $\exists\, u(t)$ s.t. $-x(0) = e^{-AT}\int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau = \int_0^T e^{-\tau}Bu(\tau)\,d\tau$
- ***Reachable*** if $x(0) = 0$ can be driven to any state $x_f = x(T)$ in time T
  - i.e. $\exists\, u(t)$ s.t. $x(T) = \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau$

General Principle: Linear independence of N *functions* $l_i(t), i = 1, \ldots, N$ over interval $[t_o, t_f]$is determined using a *Gramian:*

$$G = [G_{ij}], \qquad G_{ij} = \int_{t_0}^{t_f} l_i(\tau)l_j(\tau)\,d\tau$$

Linear independence is proven when G has full rank

Controllability

Controllability Gramian:

$$C(t_0, t_1) = \int_{t_0}^{t_f} e^{A(t_0-\tau)} B B^T e^{A^T(t_0-\tau)} d\tau \quad \rightarrow \quad C(0, t_f) = \int_0^{t_f} e^{-A\tau} B B^T e^{-A^T\tau} d\tau$$

Since $C(0, t_f)$ is symmetric, for it to be full rank over $[0, t_f]$, it must be positive definite.

Lemma: $C(0, t_f)$ is positive definite *if and only if* there is no vector $v \neq 0$ such that

$$v^T e^{-At} B = 0 \quad \forall t \epsilon [0, t_f]$$

Proof (by contradiction)**:** suppose there is such a v with $v^T e^{-At} B = 0 \quad \forall t \epsilon [0, t_f]$

- $v^T C(0, t_f) v = \int_0^{t_f} v^T e^{-A\tau} B B^T e^{-A^T\tau} B v \; d\tau$
- If there is such a v, then $v^T C(0, t_f) v = 0$, which implies that $C(0, t_f) v$ is not positive definite.

Theorem: The pair (A,B) is controllable if and only if the $C(0, t_f)$ is positive definite

Proof (sufficiency)**:** suppose $C(0, t_f)$ is positive definite. Let x_0, x_f be the initial/final states

- $x(t_f) = e^{At_f} x_0 + \int_0^{t_f} e^{-A(t_f-\tau)} B \, u(\tau) \, d\tau$

Controllability

Proof (sufficiency)**:** *(continued)*

- Choose $u(t) = B^T e^{-A^T t} C^{-1}(0, t_f)v$ for some constant vector v
- Then: $x(t_f) = e^{At_f} x_0 + \int_0^{t_f} e^{A(t_f-\tau)} B\, B\text{^}T e^{-A^T \tau} C^{-1}(0, t_f) v\, d\tau$
  $$= e^{At_f} x_0 + e^{At_f} C(0, t_f) C^{-1}(0, t_f) v$$
  $$= e^{At_f}(x_0 + v)$$
- If $v = -x_0 + e^{-At_f} x_f$, then $x(t_f) = x_f$

That is, $u(t) = B^T e^{-A^T t} C^{-1}(0, t_f)[e^{-At_f} x_f - x_0]$ steers x_0 to x_f for any x_0, x_f

Proof (necessity)**:** show that positive definiteness of $C(0, t_f)$ is necessary

- Contradiction: suppose $C(0, t_f)$ is not positive definite.
- Then there exists $z \neq 0$ such that $z^T e^{-At_f} B = 0 \quad \forall t \epsilon [0, t_f]$
- For ***controllability***, let $\mathrm{x}_0 = z$**.** Suppose that $x(t_f) = 0$
  - Then: $0 = e^{At_f} z + \int_0^{t_f} e^{A(t_f-\tau)} B\, u(\tau)\, d\tau$
  - Multiply by $z^T e^{-At_f}$: $0 = z^T z + \int_0^{t_f} z^T e^{A\tau} B\, u(\tau)\, d\tau$
  - But integrand is zero for all t, and thus $z = 0$, a contradiction

Controllability/Reachability

Proof (necessity)**:** *(continued)*

- For ***reachability***, let $\mathrm{x_f} = e^{At_f}z$, and suppose $u(t)$ steers x_0 to $x(t_f) = x_f$
  - Then: $e^{At_f}z = \int_0^{t_f} e^{A(t_f-\tau)}B\,u(\tau)\,d\tau$
  - Multiply by $z^Te^{-At_f}$: $\quad z^Te^{-At_f}e^{At_f}z = \int_0^{t_f} z^Te^{-A\tau}B\,u(\tau)d\tau = z^Tz$
  - But, if $C(0,t_f)$ is not positive definite, then there exists z such that $z^Te^{-At_f}B = 0 \quad \forall t\epsilon[0,t_f]$, implying that $z = 0$, which is a is a contradiction.

Theorem: $C(0,t_f)$ is positive definite only if $rank(W_r) = n$, where

$$W_r = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

Proof: If $C(0,t_f)$ is not positive definite, there exists $z \neq 0$ s. t. $z^Te^{-At_f}B = 0, \forall t\epsilon[0,t_f]$

- $z^T\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}A^kB = 0, \forall t\epsilon[0,t_f]$
- Same as $\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}z^TA^kB = 0, \forall t\epsilon[0,t_f]$
- This implies that there exists z such that $z^TA^kB = 0$ for all $k = 0,1,\ldots$

Controllability/Reachability/Observability

Proof: (*continued)*

- $\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} z^T A^k B = 0$, $\forall t \epsilon [0, t_f]$ implies via Cayley-Hamilton that

$$z^T A^k B = 0 \text{ for } k = 0, \dots, n-1$$

- Hence, $z^T \left[B \; AB \; A^2 B \cdots A^{n-1} B \right] = 0$, which implies that W_r is not full rank.
- Therefore, (A,B) is reachable (controllable) only if W_r is full rank n

Note: in LTI case, reachability is independent of time.

Observability Gramian:

$$O(0, t_f) = \int_0^{t_f} e^{-A^T \tau} C^T C e^{-A\tau} d\tau$$

A nearly identical analysis shows that the O must be positive definite for observability, which in turn implies that the observability matrix W_O must be full rank.

Mid Term

Schedule: (1) Handed out in Class on Monday. (2) Due Friday at 5:00 pm.

Instructions on Front Page. Three hour limited time take-home.

Review:

- Convert control system description to 1^{st} order form
- Solution and characterization of o.d.e.s
    - Matrix exponential, equilibria, stability of equilibria, phase space
- Lyapunov Function and stability
- System linearization, and stability/stabilization of linearized models.
- Convolution Integral, impulse response
- Performance characterization for 1^{st} and 2^{nd} order systems:
    - Step response overshoot, rise time, settling time
- System Frequency Response
- Discrete Time System
- State Feedback, eigenvalue placement
- Reachability, reachable canonical form, test for reachability