CDS 101/110: Lecture 10.3
Final Exam Review

December 2, 2016

Schedule:

(1) Posted on the web Monday, Dec. 5 by noon.

(2) Due Friday, Dec. 9, at 5:00 pm.

(3) Determines 30% of your grade

Instructions on Front Page.

- Five hour limited time take-home.
- Same collaboration rules as Mid-Term

Key Concepts up to Mid Term

Review:

- Frequency domain Convert control system description to 1st order form
- Solution and characterization of o.d.e.s
  - Matrix exponential, equilibria, stability of equilibria, phase space
- Lyapunov Function and stability
- System linearization, and stability/stabilization of linearized models.
- Convolution Integral, impulse response
- Performance characterization for 1st and 2nd order systems:
  - Step response overshoot, rise time, settling time
- System Frequency Response
- Discrete Time System
- State Feedback, eigenvalue placement
- Reachability, reachable canonical form, test for reachability

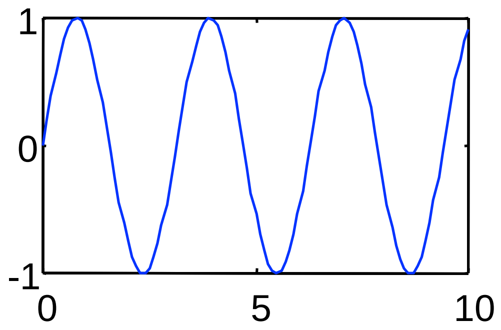
Key Concepts From Mid-Term Onward

Review:

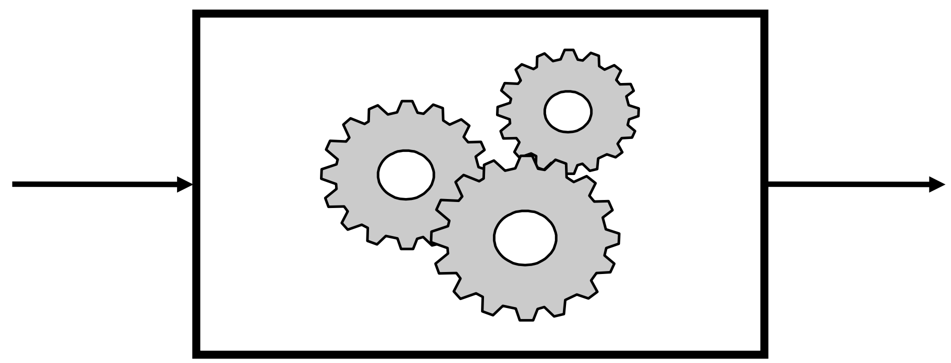
- Frequency Domain Concepts
  - Transfer Function (poles/zeros)
  - Block Diagram Algebra
  - Bode Plot
- Loop Diagram Concepts
  - Loop Transfer Function (closed loop poles and zeros)
  - Nyquist Plot and Nyquist Criterion for closed loop stability
  - Gain, Phase, and Stability Margins
- PID Controllers
  - Effect of “P”, “I”, and “D” terms of closed loop behavior
  - Reachability, reachable canonical form, test for reachability
- Loop Shaping
  - Lead/Lag compensators
  - Converting requirements/spec.s to frequency domain equivalents
  - Sensitivity Functions (“gang of four”)

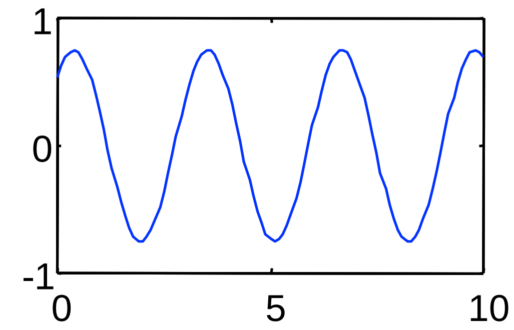
Frequency Domain Modeling

Defn. The frequency response of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.

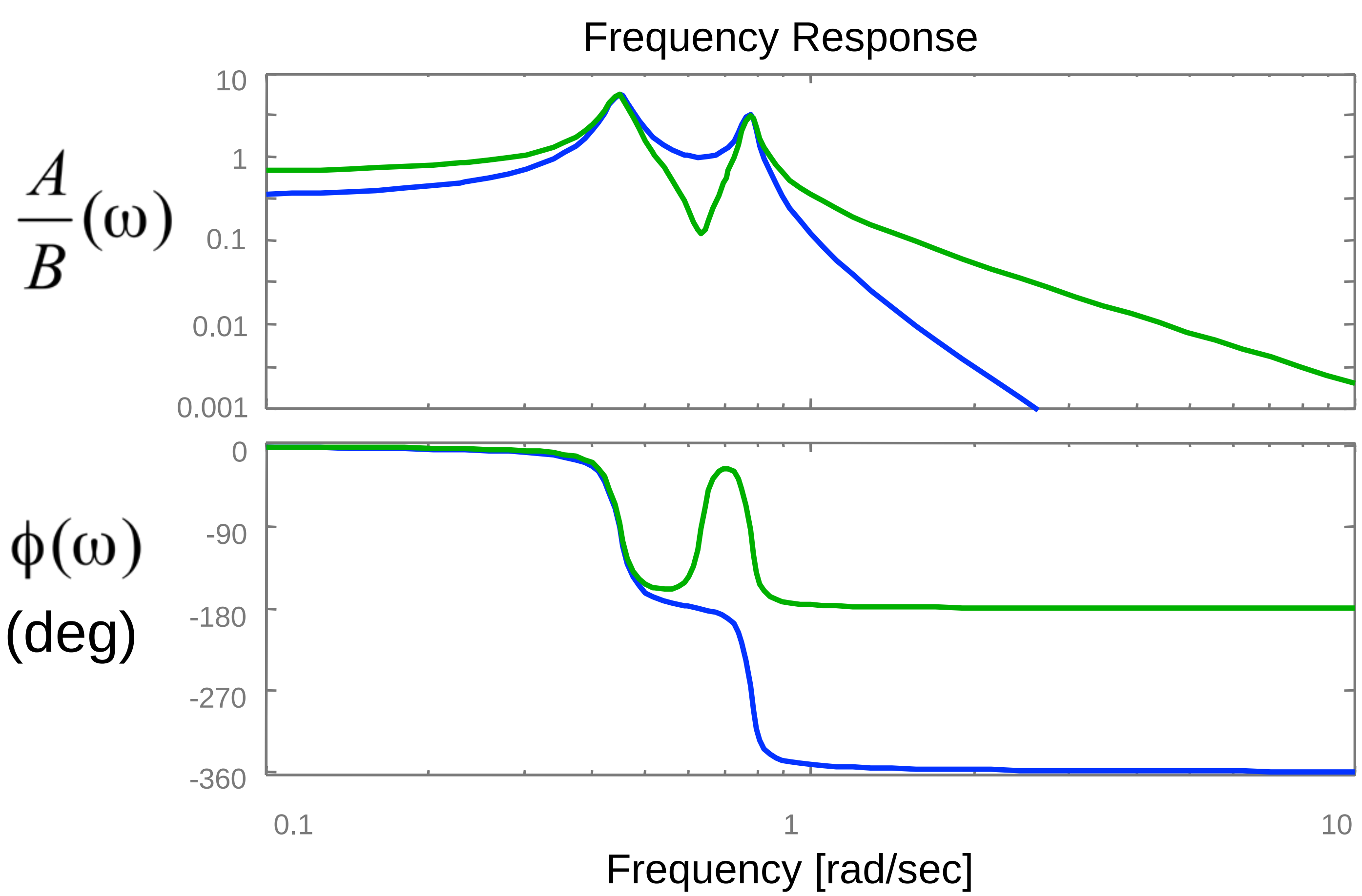


$u = A\sin(\omega t)$





$y = B\sin(\omega t + \phi)$



Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity ⇒ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

Transfer Function Properties

$$y(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \Big(C(sI - A)^{-1}B + D\Big)e^{st}$$

transient steady state

Theorem. The transfer function for a linear system $\Sigma = (A, B, C, D)$ is given by

$$G(s) = C(sI - A)^{-1} + D \qquad s \in \mathbb{C}$$

Theorem. The transfer function $G(s)$ has the following properties (for SISO systems):

- G(s) is a ratio of polynomials n(s)/d(s) where d(s) is the characteristic equation for the matrix A and n(s) has order less than or equal to d(s).
- The steady state frequency response of Σ has gain |G(jω)| and phase arg G(jω):

$$u = Msin(\omega t)$$

$$y = |G(i\omega)|Msin(\omega t + \arg G(i\omega)) + transients$$

Remarks

- G(s) is the Laplace transform of the impulse response of Σ
- Typically we write “y = G(s)u” for Y(s) = G(s)U(s), where Y(s) & U(s) are Laplace transforms of y(t) and u(t).
- MATLAB: G = ss2tf(A, B, C, D)

Laplace Transform Review

Constant Coefficient O.D.E.: Laplace Transform (assuming zero initial conditions)

$$\frac{d^n}{dt^n}y(t) + a_1\frac{d^{n-1}}{dt^{n-1}}y(t) + \cdots + a_n y(t) = b_1\frac{d^{n-1}}{dt^{n-1}}u(t) + \cdots + b_n u(t) \quad (*)$$

$\mathcal{L}\{\cdot\}$

$$\left(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\right)Y(s) = \left(b_1 s^{n-1} + \cdots + b_n\right)U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\left(b_1 s^{n-1} + \cdots + b_n\right)}{\left(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\right)} = \frac{n(s)}{d(s)}$$

- Roots of $d(s)$ are called the *poles* of transfer function $G(s)$
  - If p is a system pole, then $y = e^{pt}$ is a solution to (*) with $u(t) = 0$
  - Poles are strictly defined by matrix A..
- Roots of $n(s)$ are called the *zeros* of G(s)
  - If s is a pole of $G(s)$, then $G(s)e^{st}$ is an output if $d(s) \neq 0$.
  - Out put is zero at s if $n(s) = 0$.

Poles and Zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

- Roots of $d(s)$ are called *poles* of $G(s)$
- Roots of $n(s)$ are called *zeros* of $G(s)$

Poles of G(s) determine the stability of the (closed loop) system

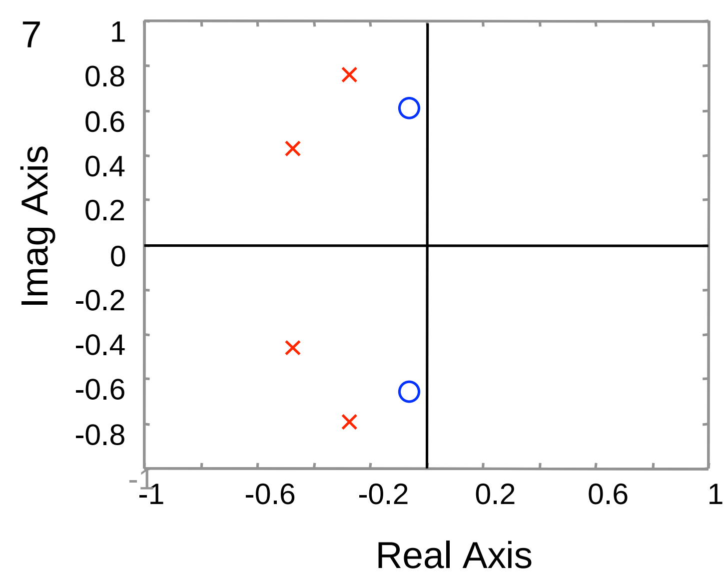
- Denominator of transfer function = characteristic polynomial of state space system
- Provides easy method for computing stability of systems
- Right half plane (RHP) poles (Re > 0) correspond to unstable systems

Zeros of G(s) related to frequency ranges with limited transmission

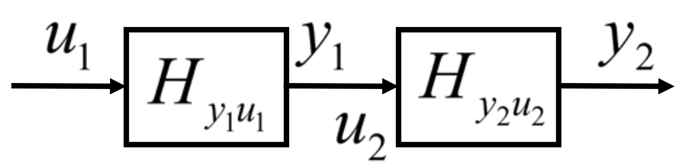
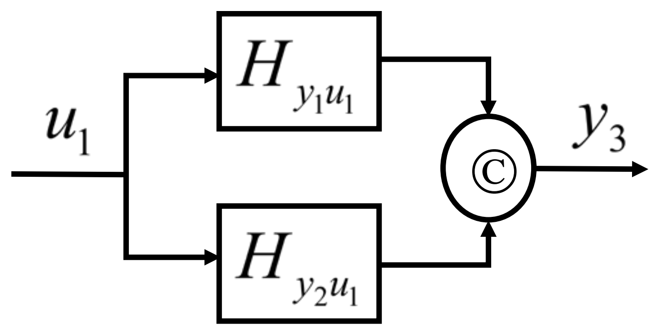
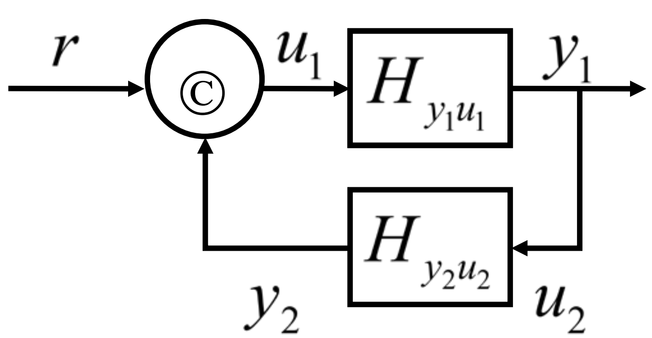
- A pure imaginary zero at s = iω blocks any output at that frequency (G(iω) = 0)
- Zeros provide limits on performance, especially RHP zeros

MATLAB: pole(G), zero (G), pzmap(G)

$$G(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \xrightarrow{\text{pzmap(G)}}$$



Block Diagram Algebra

Type	Diagram	Transfer function
Series		$H_{y_2u_1} = H_{y_2u_2}H_{y_1u_1} = \frac{n_1n_2}{d_1d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \frac{n_1d_2 + n_2d_1}{d_1d_2}$
Feedback		$H_{y_1r} = \frac{H_{y_1u_1}}{1 + H_{y_1u_1}H_{y_2u_2}} = \frac{n_1d_2}{n_1n_2 + d_1d_2}$

- These are the basic manipulations needed; some others are possible
- Formally, could work all of this out using the original ODEs (⇒ nothing really new)

Sketching the Bode Plot for a Transfer Function (1/2)

Evaluate transfer function on imaginary axis

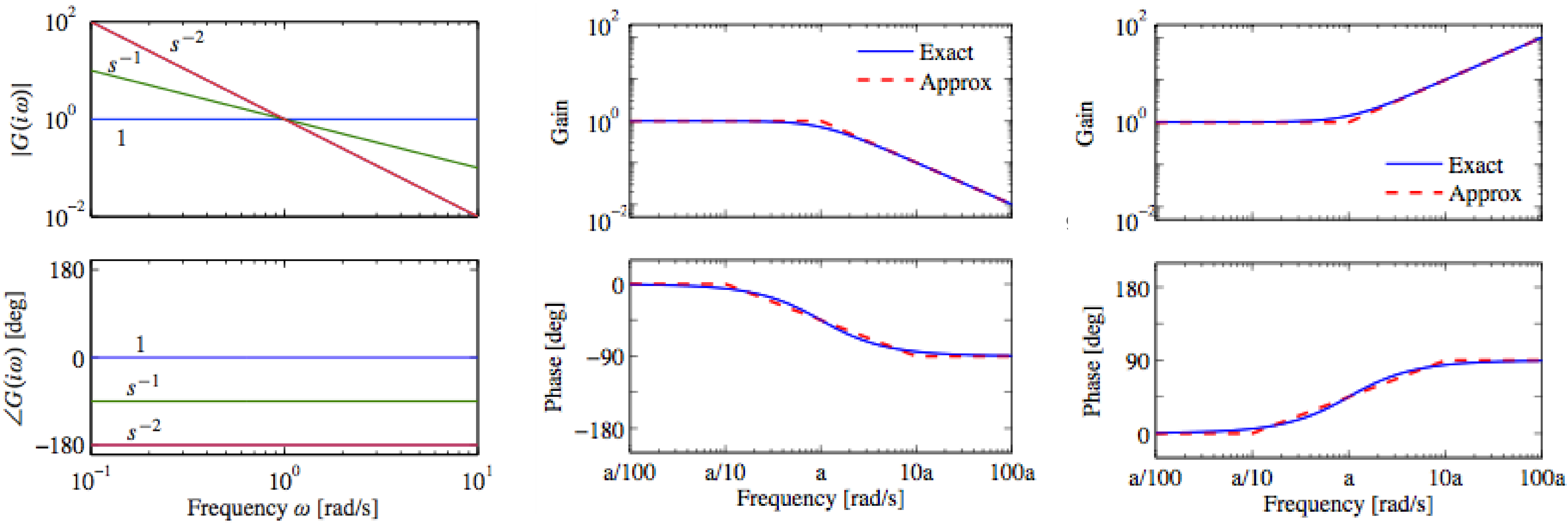
$$M = |G(i\omega)|, \qquad \varphi = \arctan \frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}$$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a \\ \log a - \log \omega & \text{if } \omega > a, \end{cases}$$

- Plot gain (M) on log/log scale
- Plot phase (φ) on log/linear scale
- Piecewise linear approximations available

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10 \\ -45 - 45(\log\omega - \log a) & a/10 < \omega < 10a \\ -90 & \text{if } \omega > 10a. \end{cases}$$

$$G(s) = \frac{1}{s^k} \qquad G(s) = \frac{a}{s+a} \qquad G(s) = \frac{s+a}{a}$$



Sketching the Bode Plot for a Transfer Function (2/2)

Complex poles $G(s) = \dfrac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0, \end{cases}$$

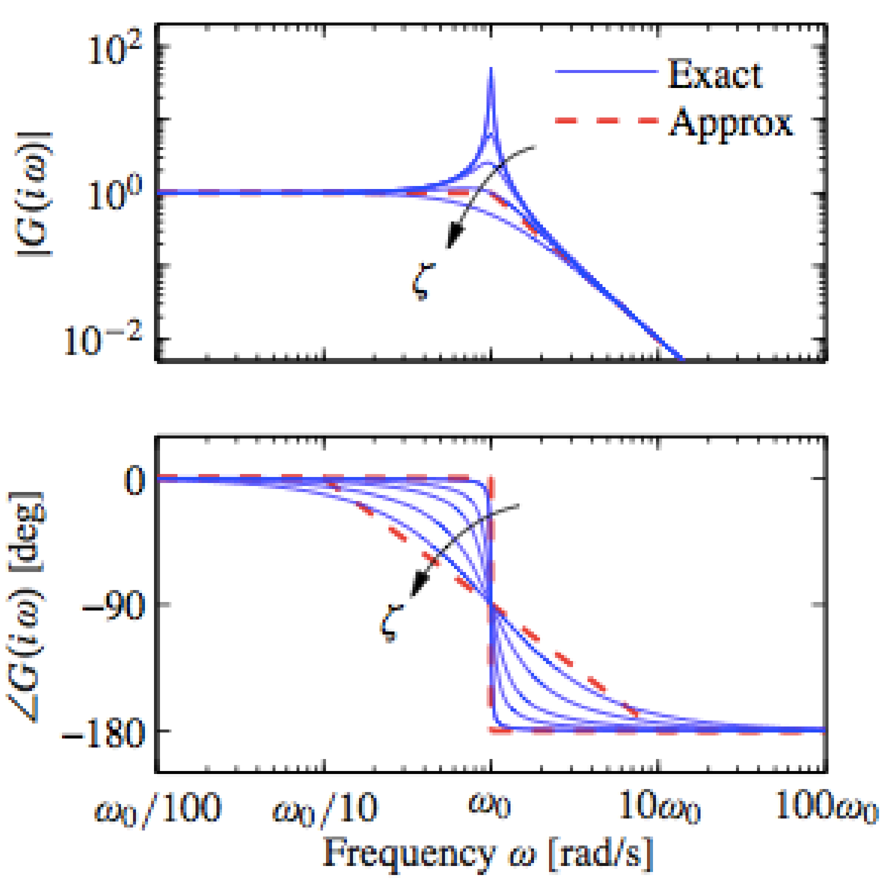
$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}$$

Ratios of products $G(s) = \dfrac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$

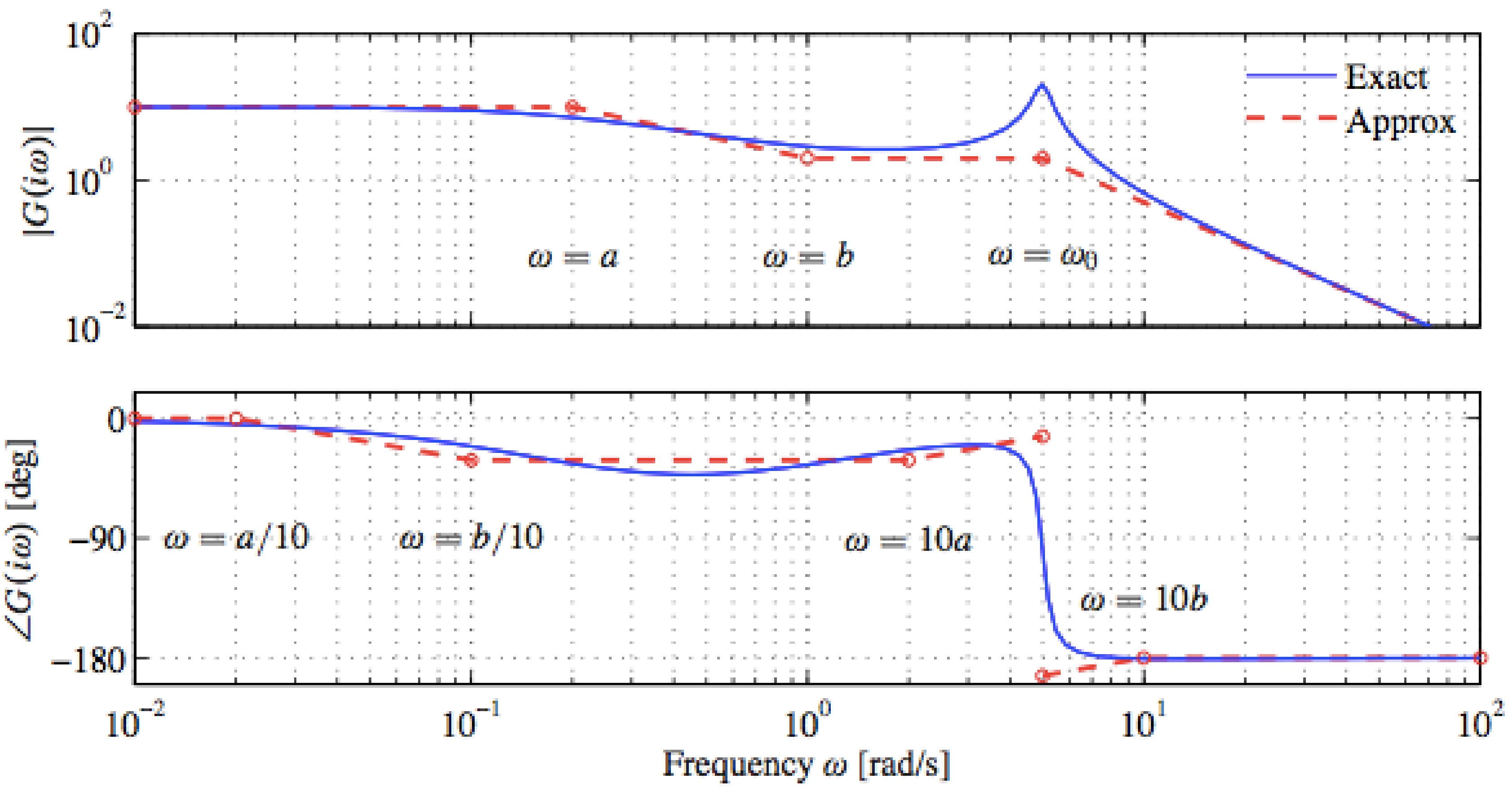
$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|$$

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s)$$

$$G(s) = \frac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$$



$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \qquad a \ll b \ll \omega_0.$$



Bode Plot Units

What are the units of a Bode Plot?

- **Magnitude:** The ordinate (or “y-axis”) of magnitude plot is determined by $20\log_{10}|G(i\omega)|$
  - Decibels,” names after A.G. Bell
- **Phase:** Ordinate has units of degrees (of phase shift)
- The abscissa (or “x-axis”) is $\log_{10}(\text{frequency})$ (usually, rad/sec)

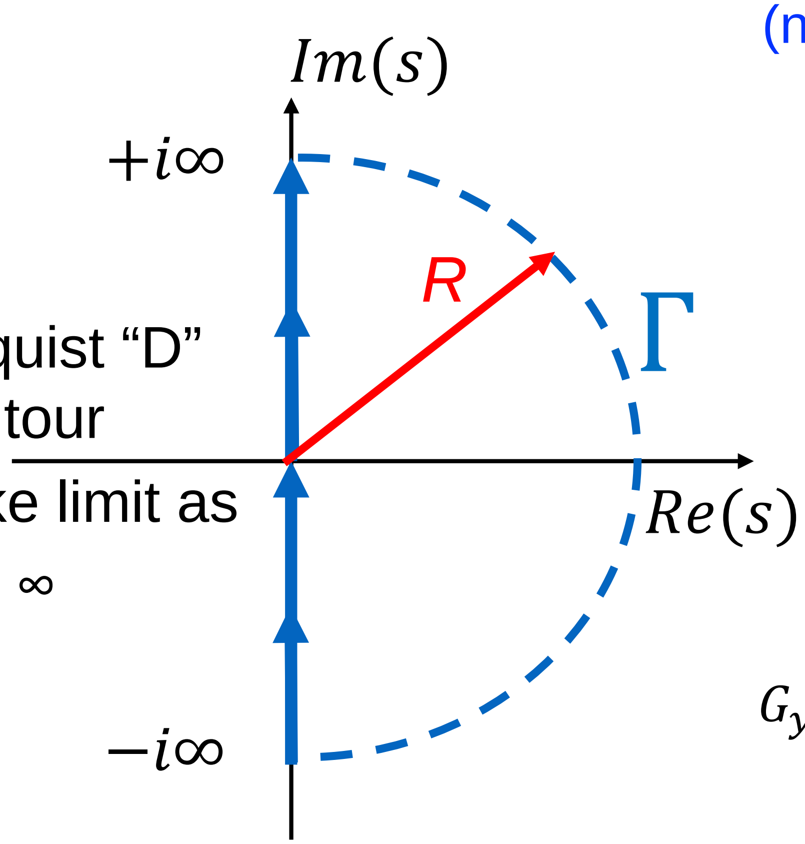
Example: simple first order system: $G(s) = \frac{1}{1+\tau s}$

- Single pole at $s = -1/\tau$
- $|G(i\omega)| = \left|\frac{1}{1+i\tau\omega}\right| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$
- In decibels:

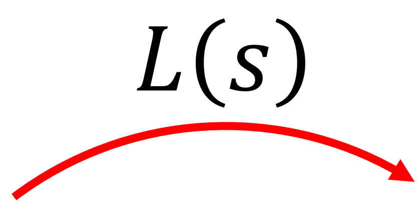
$$
\begin{aligned}
20\log_{10}|G(i\omega)| &= 20\log_{10}1 - 20\log_{10}\left(1+(\omega\tau)^2\right)^{\frac{1}{2}} \\
&= -10\log_{10}\left(1+(\omega\tau)^2\right)
\end{aligned}
$$

Basic Nyquist Plot (review)

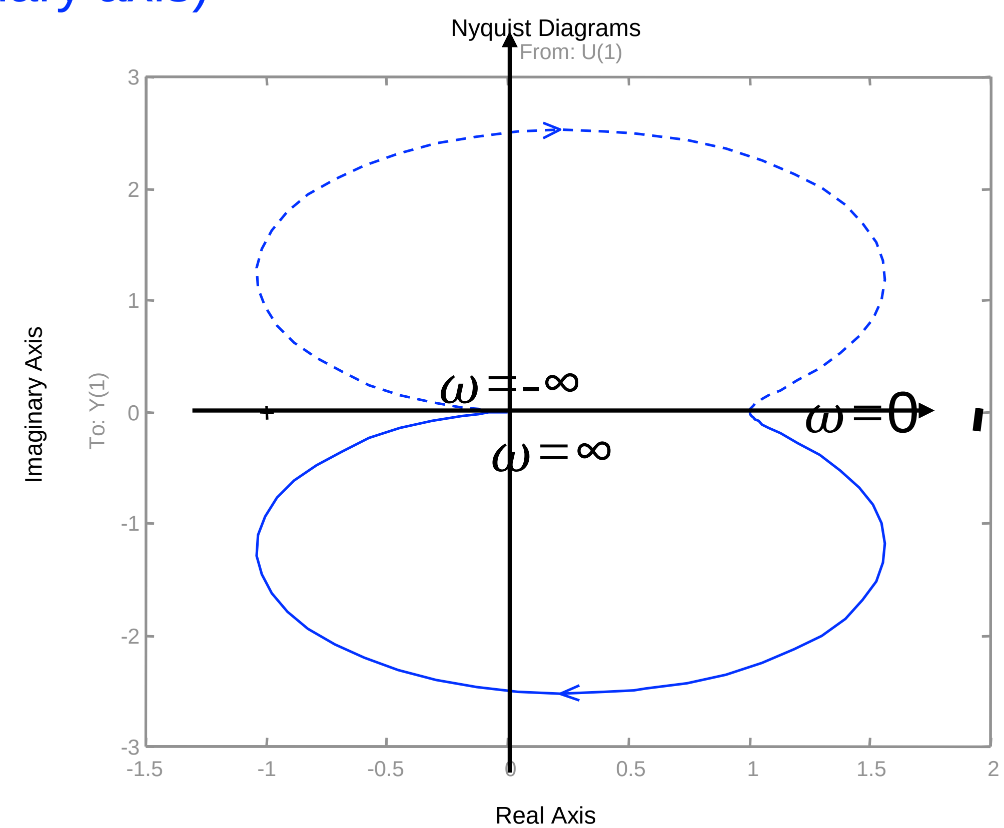
(no poles of $L(s)$ on imaginary axis)



- Nyquist "D" contour
- Take limit as $R \to \infty$



$$G_{yr}(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



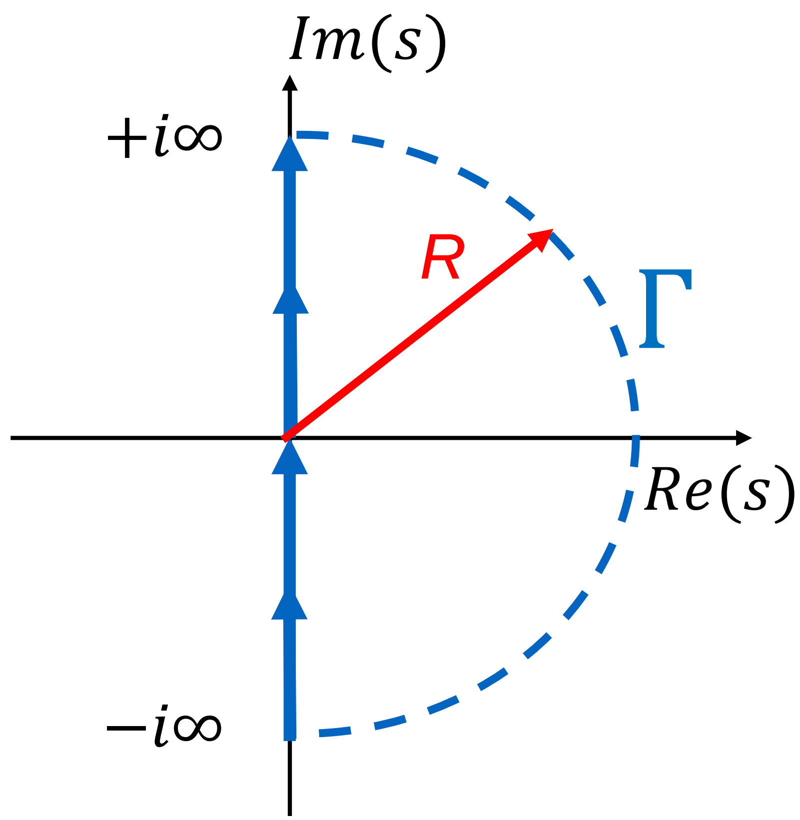
Nyquist Contour (Γ)**:**

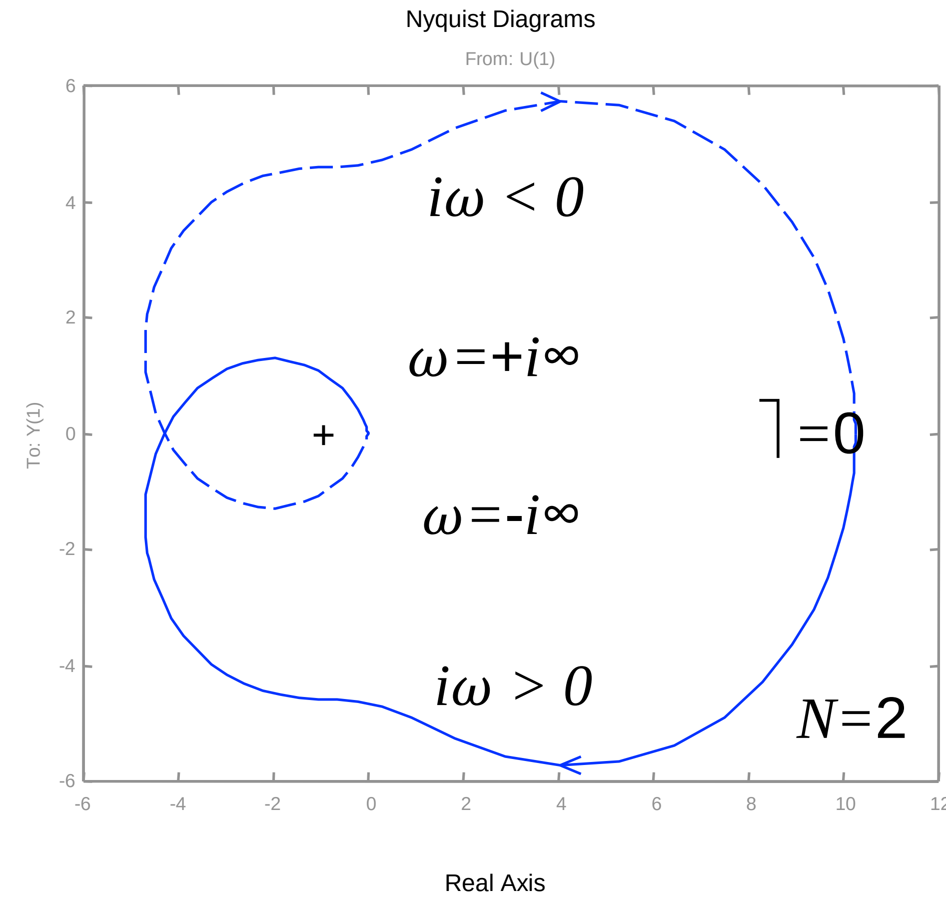
- Start from 0, and move along positive Imaginary axis (increasing frequency)
- Follow semi-Circle, or arc at infinity, in clockwise direction (connecting the endpoints of the imaginary axis)
- From $-i\infty$ to zero on imaginary axis
- Note, portion of plot corresponding to $\omega < 0$ is mirror image of $\omega > 0$

Nyquist Plot

- Formed by tracing s around the Nyquist contour, Γ, and mapping through $L(s)$ to complex plane representing magnitude and phase of $L(s)$.
- I.e., the image of $L(s)$ as s traverses Γ is the Nyquist plot
- **Goal:** from complex analysis, we're trying to find number of zeros (if any) in RHP, which leads to instability

Nyquist Criterion





Thm (Nyquist). Consider the Nyquist plot for loop transfer function *L*(*s*). Let

P # RHP poles of open loop $L(s)$

N # clockwise encirclements of -1 (counterclockwise is negative)

Z # RHP zeros of 1 + $L(s)$

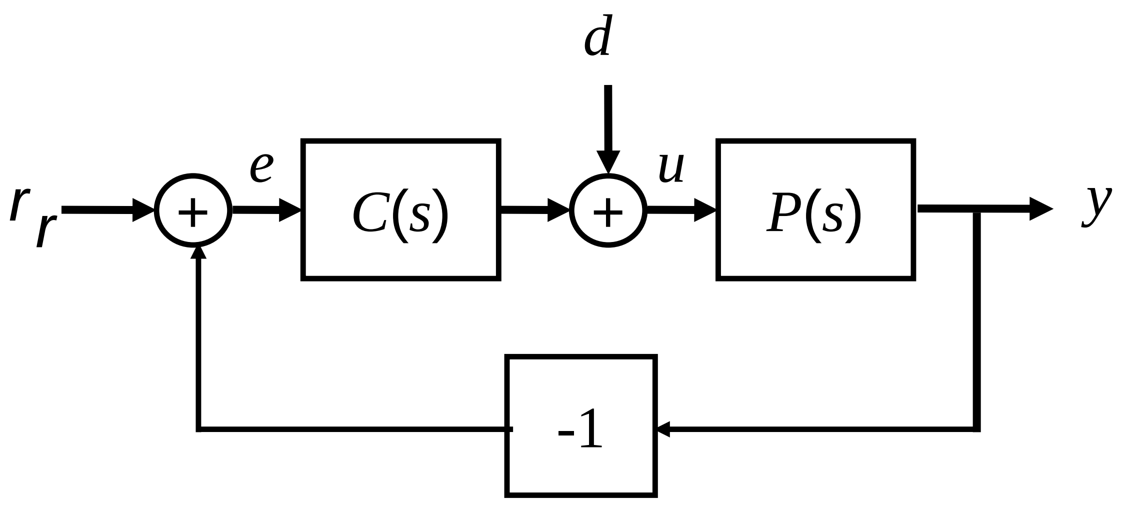
Then

$$Z = N + P$$

Consequence:

- If $Z \geq 1$, then $(1 + L(s))$ has RHP zeros, which means that $G_{yr}(s)$ has RHP poles.
- $G_{yr}(s)$ is unstable with simple unity feedback, and control $C(s)$

What can you do with a Nyquist Analysis?



Set Up (somewhat artificial)**:**

- **Given:** $P(s)$
  - (any unstable roots known)
- **Given:** $C(s)$
  - (any unstable roots known)
- **Q:** can negative output feedback stabilize the system (stable $G_{yr}(s)$)?

Possible Solutions:

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)}$$

- Compute and check poles of G_{yr}
- Find another way to determine existence of unstable poles without computing roots of

$$d_P(s)d_C(s) + n_P(s)n_C(s)$$

The Nyquist plot *logic*

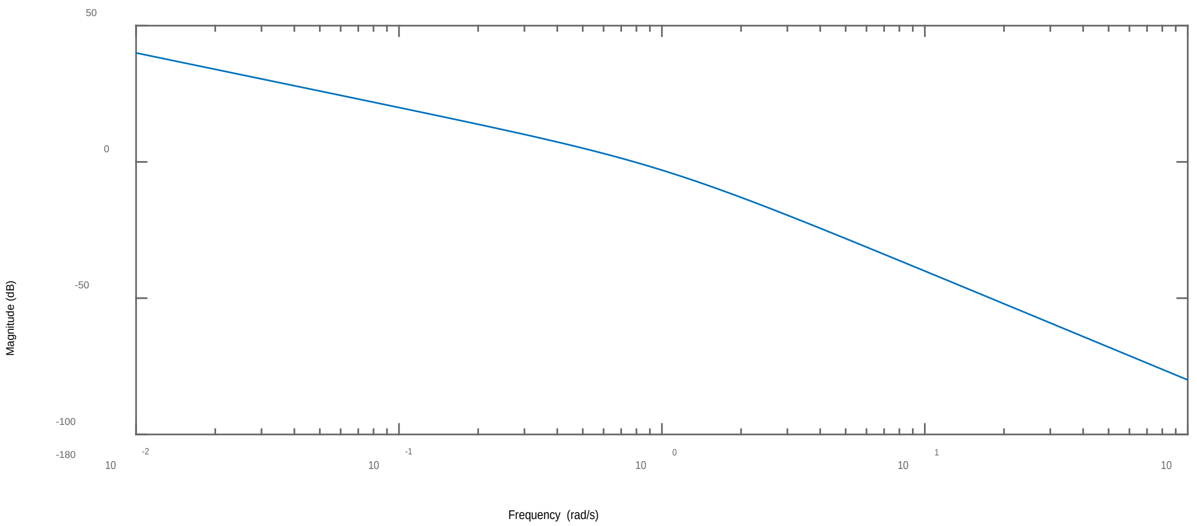
- Poles of $G_{yr}(s)$ are zeros of

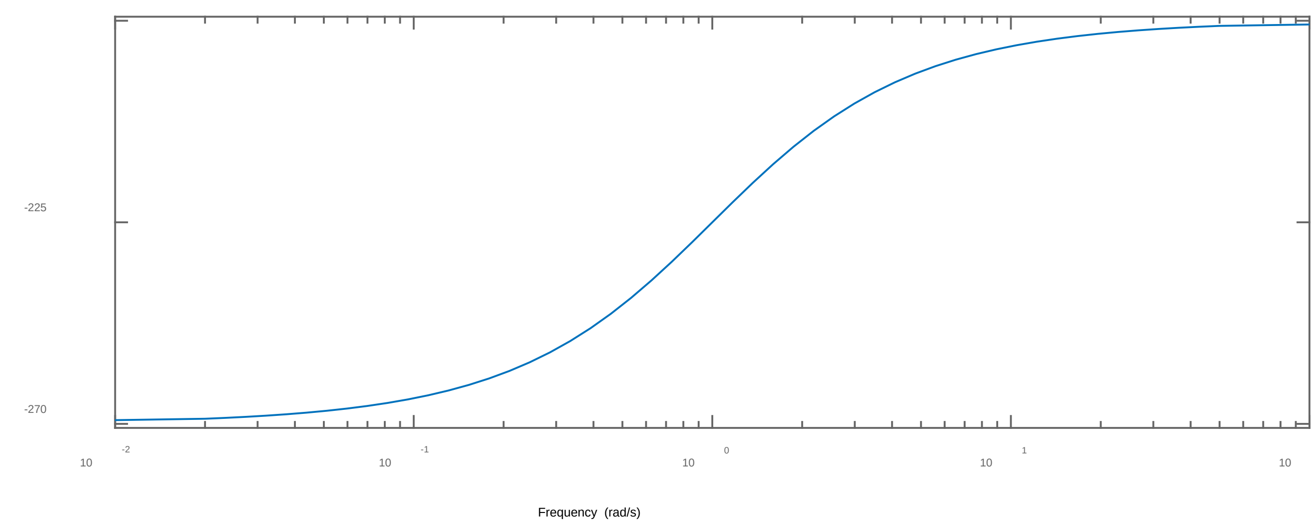
$$1 + P(s)C(s) = \frac{d_P(s)d_C(s) + n_P(s)n_C(s)}{d_P(s)d_C(s)}$$

- If $G_{yr}(s)$ is unstable, then it has at least one pole in RHP
- An unstable pole of $G_{yr}(s)$ implies and unstable (RHP) zero of $1 + P(s)C(s)$
- Nyquist plot and Nyquist Criterion allow us to determine if $1 + PC$ has RHP zeros *without* polynomial solving.

Nyquist Example (unstable system)

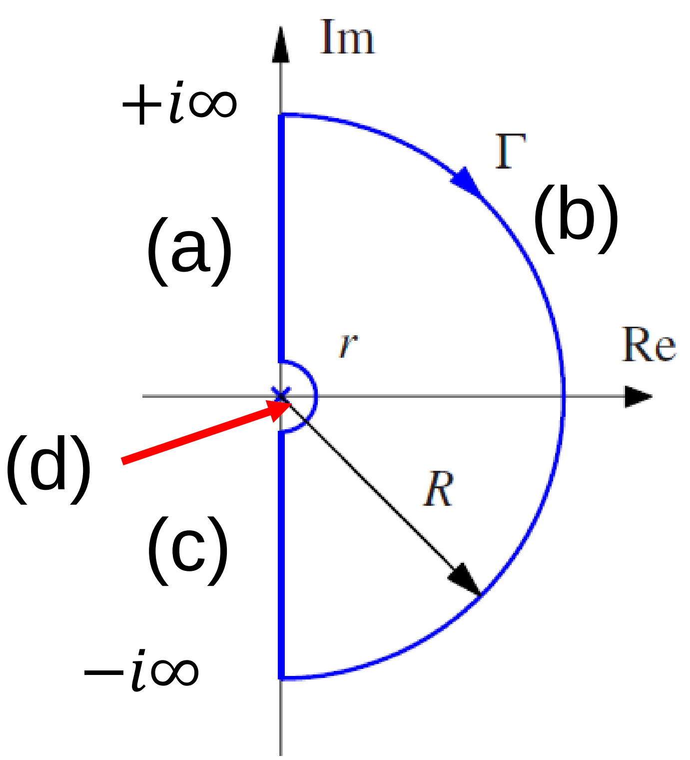
Bode Plots of Open Loop $L(s) = P(s)C(s) = \frac{k}{s(s-1)}$





Nyquist Contour and Plot

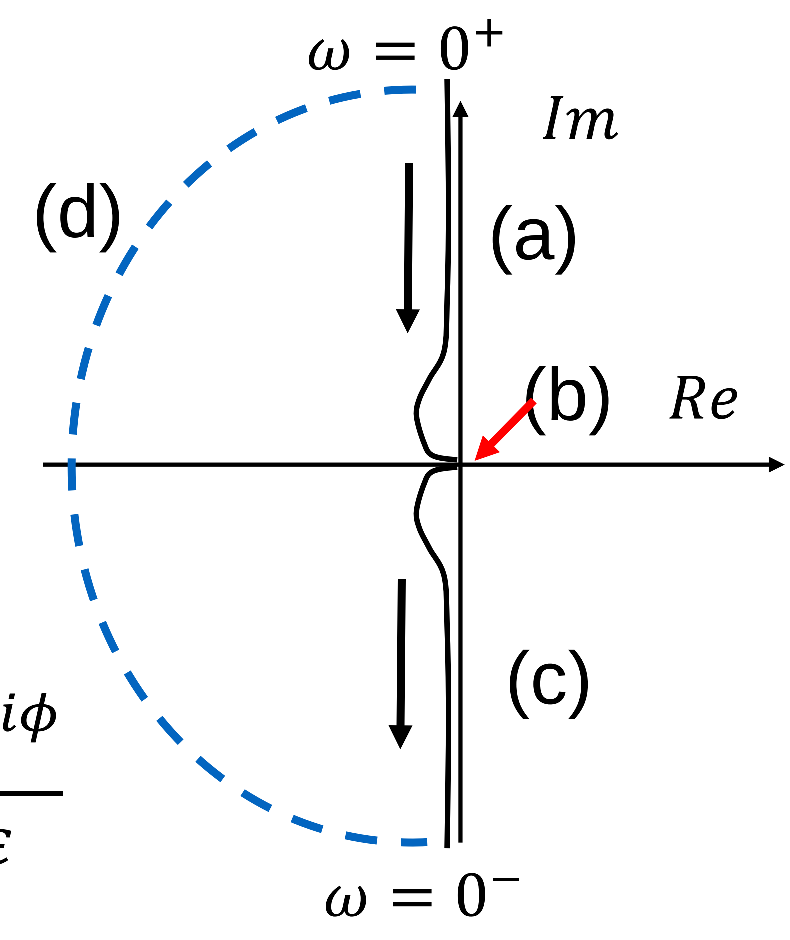
- Must account for pole on the $i\omega$ axis



$a)\ \omega = 0^+ \to +\infty$
$b)\ \omega = +\infty \to -\infty$
$c)\ \omega = -\infty \to \omega = 0^-$
$d)\ \omega = 0^- \to \omega = 0^+$

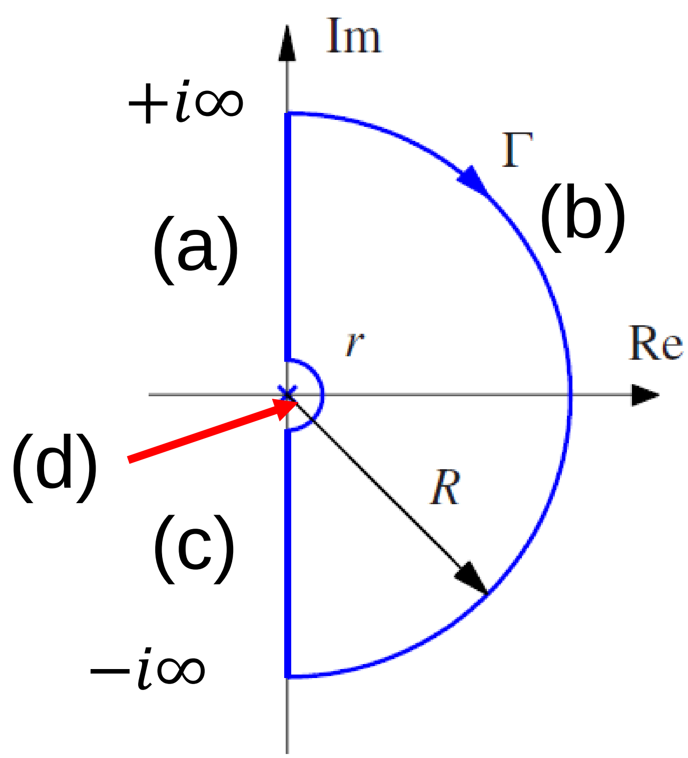
$\omega = \varepsilon e^{i\phi}$ for $[-90^o, 90^o]$

$$G\left(s = \varepsilon e^{i\phi}\right) \approx \frac{k}{-\varepsilon e^{i\phi}} = \frac{k e^{-i\phi}}{-\varepsilon}$$



Nyquist Example (unstable system)

Nyquist Contour and Plot

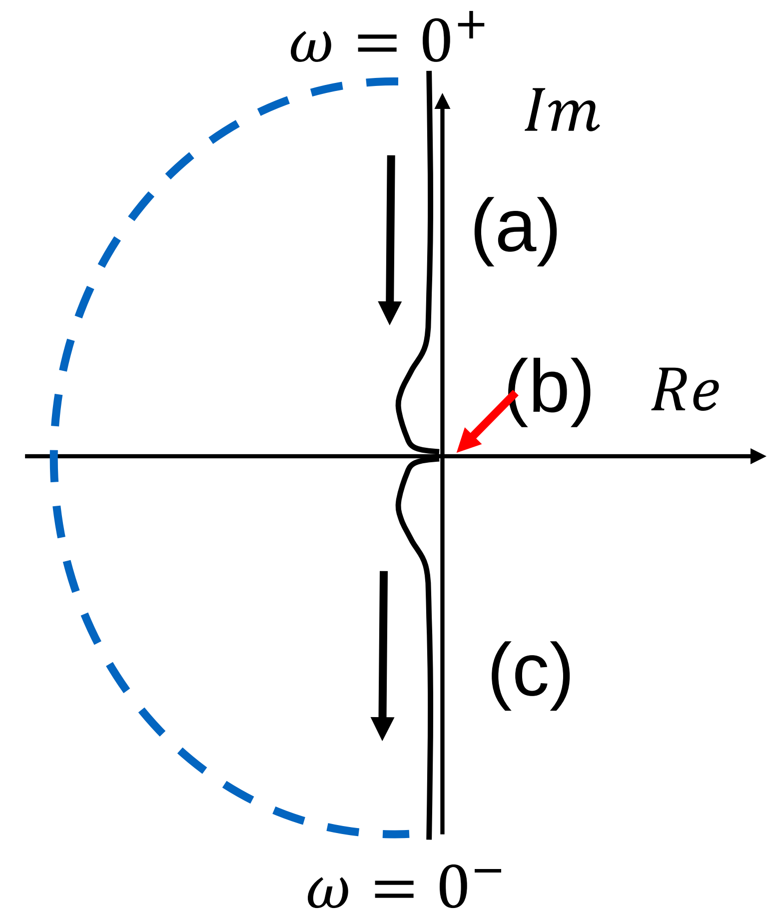


$$(d)\ \omega = 0^- \to \omega = 0^+$$

$\omega = \varepsilon e^{i\phi}$ for $[-90^o, 90^o]$

$$G\left(s = \varepsilon e^{i\phi}\right) \approx \frac{k}{-\varepsilon e^{i\phi}}$$

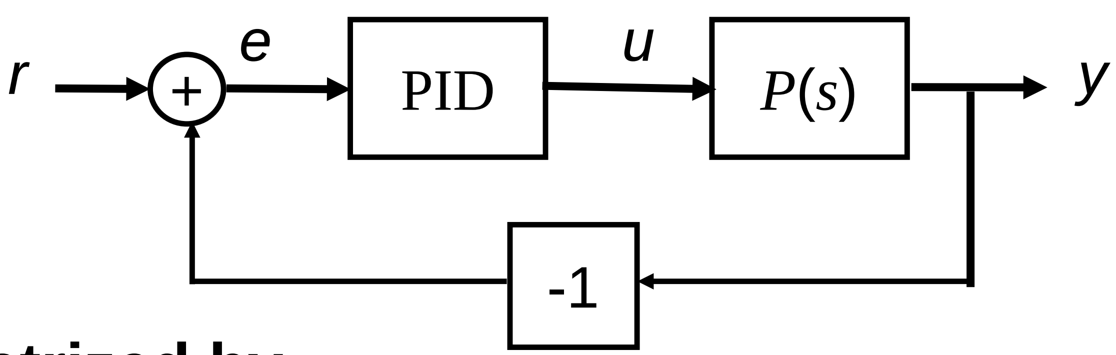
$$= \frac{ke^{-i\phi}}{-\varepsilon} = \frac{k}{\varepsilon}(-\cos\phi + i\sin\phi)$$



Accounting:

- One open loop pole in RHP: $P = 1$
- One clockwise encirclement of -1 point: $N = 1$
- $Z = N + P = 1 + 1 = 2 \quad \Rightarrow$ two unstable poles in closed loop system

Overview: PID control



$$u = k_p e + k_i \int_0^t e(\tau)\,d\tau + k_d \frac{de}{dt}$$

$$= k_p \left(e + \frac{1}{T_i} \int_0^t e(\tau)\,d\tau + T_d \frac{de}{dt} \right)$$

Parametrized by:

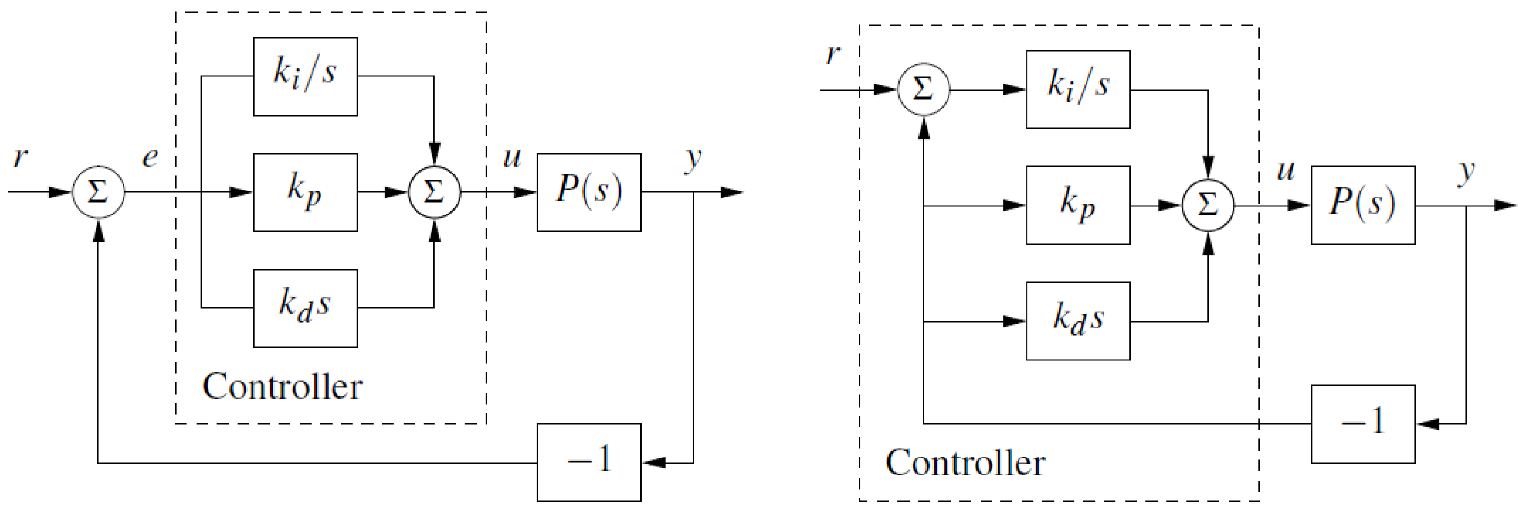
- k_p, the “proportional gain”
- k_i, the “integral gain”
- k_d, the “derivative gain”

Alternatively:

$$k_p, \quad T_i = \frac{k_p}{k_i}, \quad T_d = \frac{k_d}{k_p}$$

Utility of PID

- PID control is most common feedback structure in engineering systems
- For many systems, only need PI or PD (special case)
- Many tools for tuning PID loops and designing gains

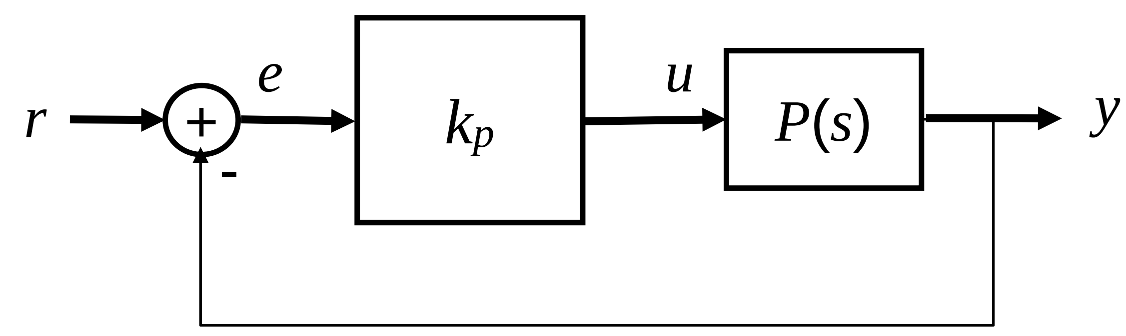


(a) PID using error feedback (b) PID using two degrees of freedom

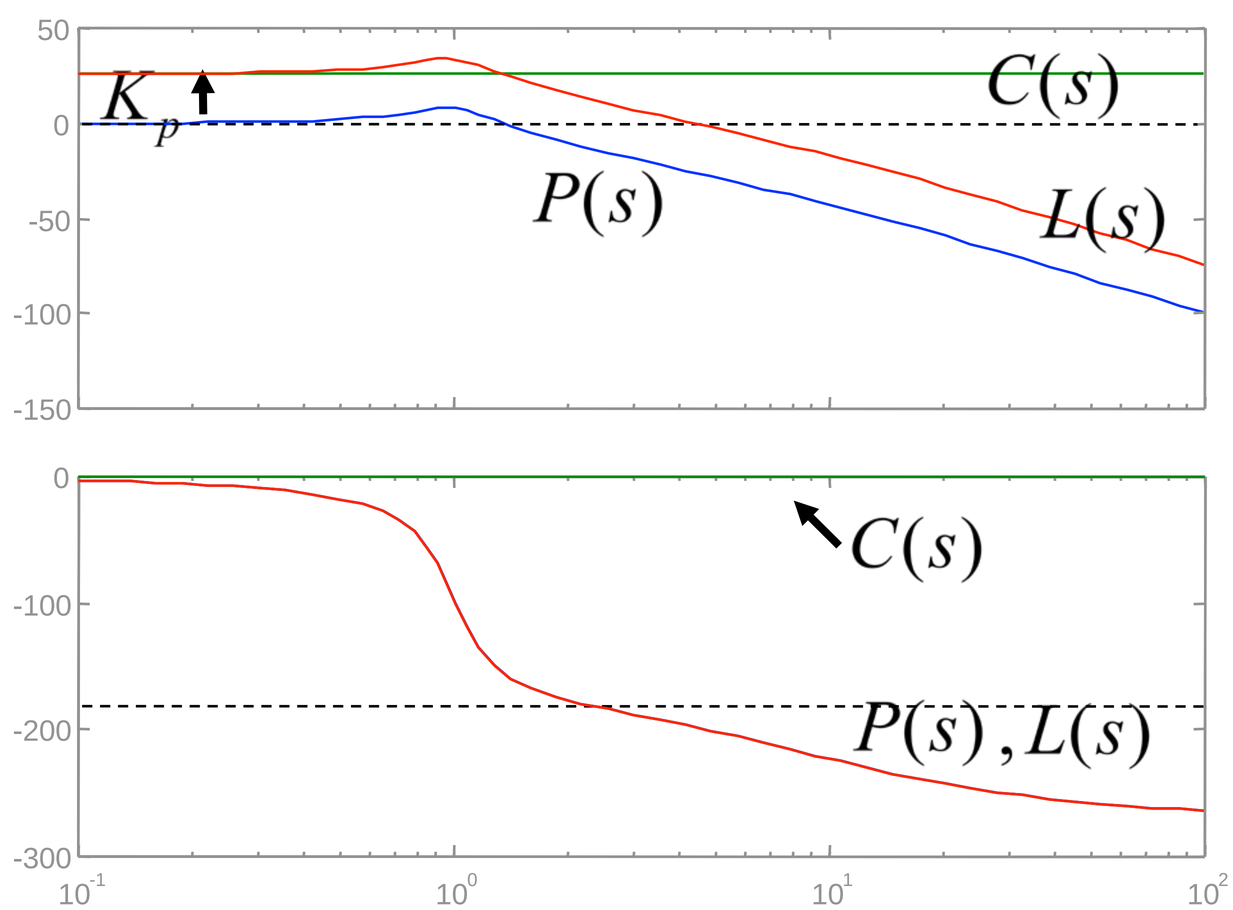
Proportional Feedback

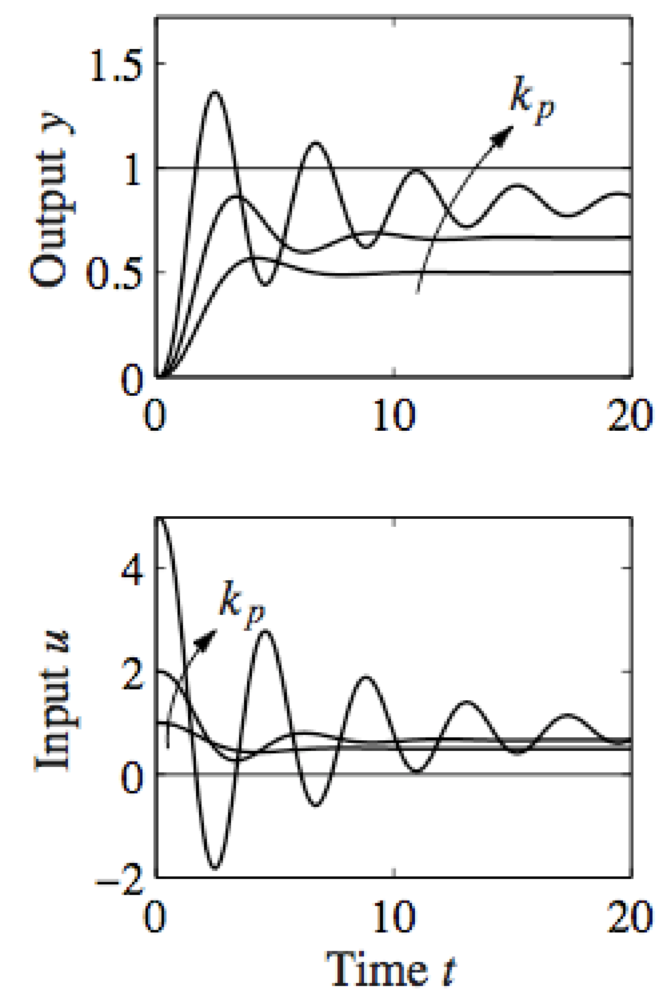
Simplest controller choice: $u = k_p e$

- Effect: lifts gain with no change in phase
- Good for plants with low phase up to desired bandwidth
- Bode: shift gain up by factor of kp
- Step response: better steady state error, but with decreasing stability



$k_p > 0$

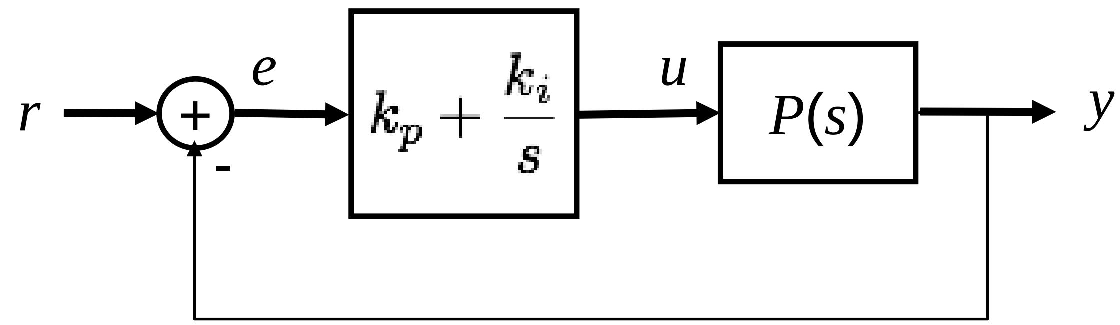




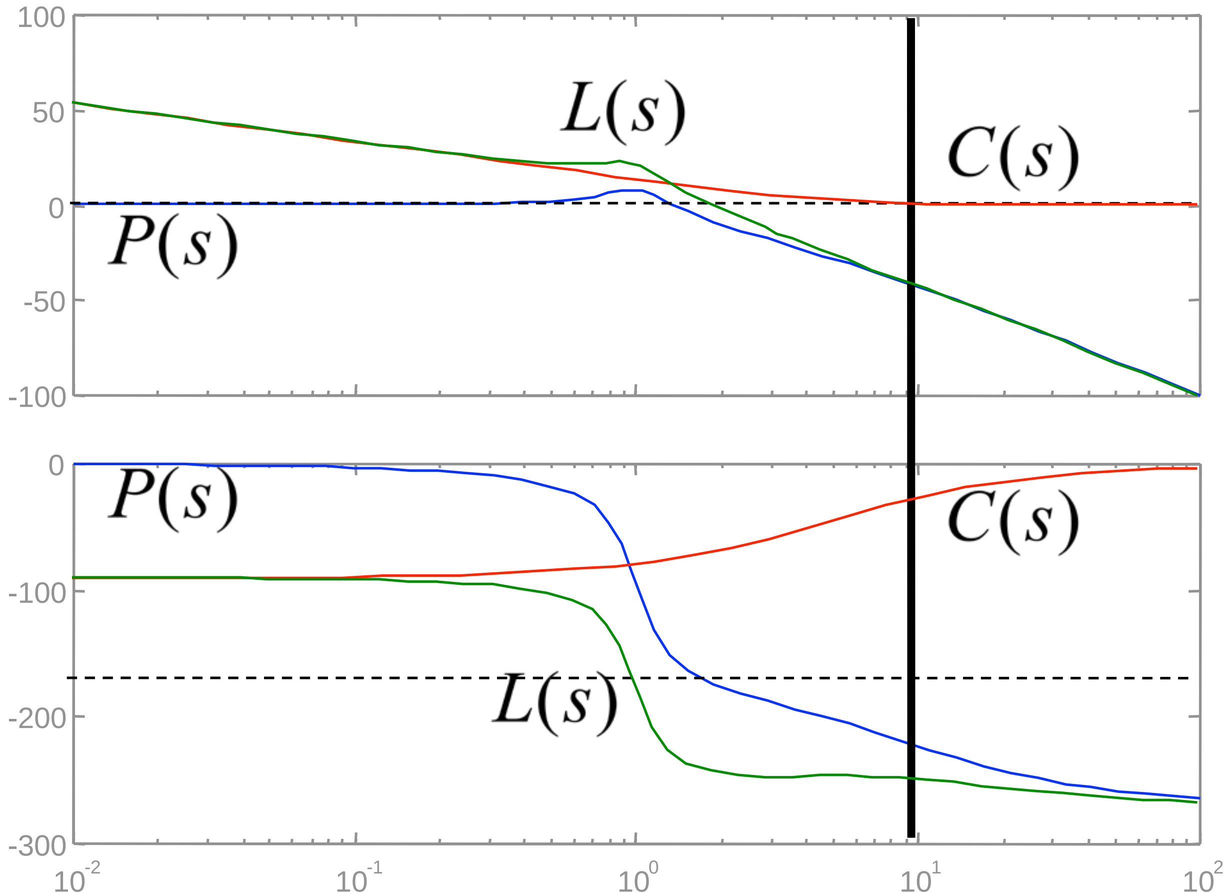
Proportional + Integral Compensation

Use to eliminate steady state error

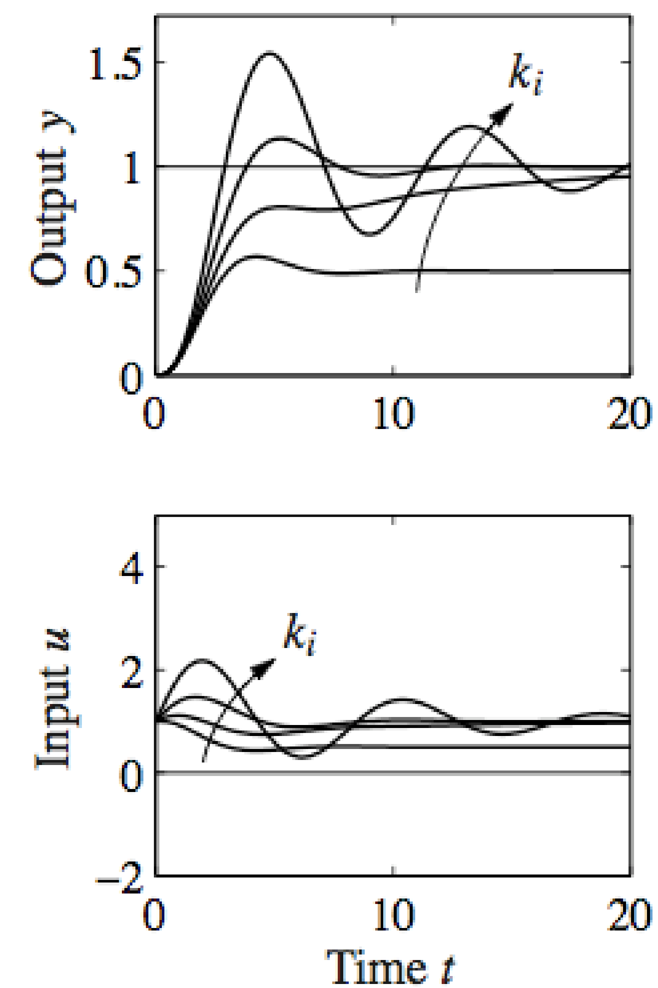
- Effect: lifts gain at low frequency
- Gives zero steady state error
- Bode: infinite SS gain + phase lag
- Step response: zero steady state error, with smaller settling time, but more overshoot



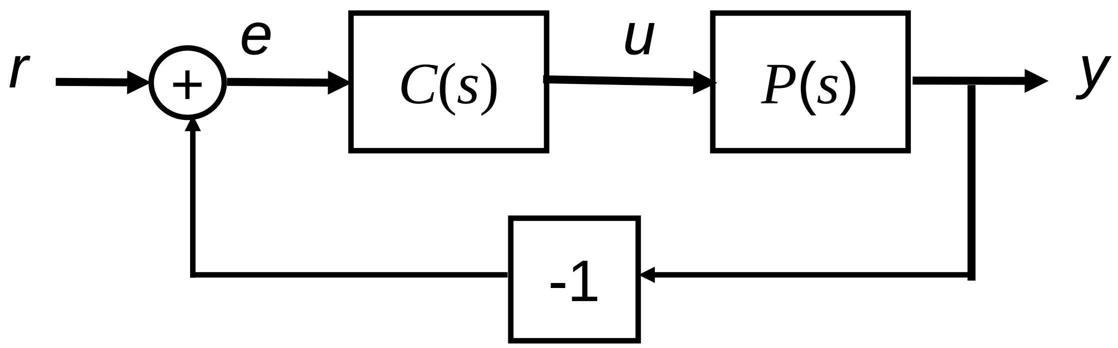
$$k_p > 0, \quad k_i > 0$$



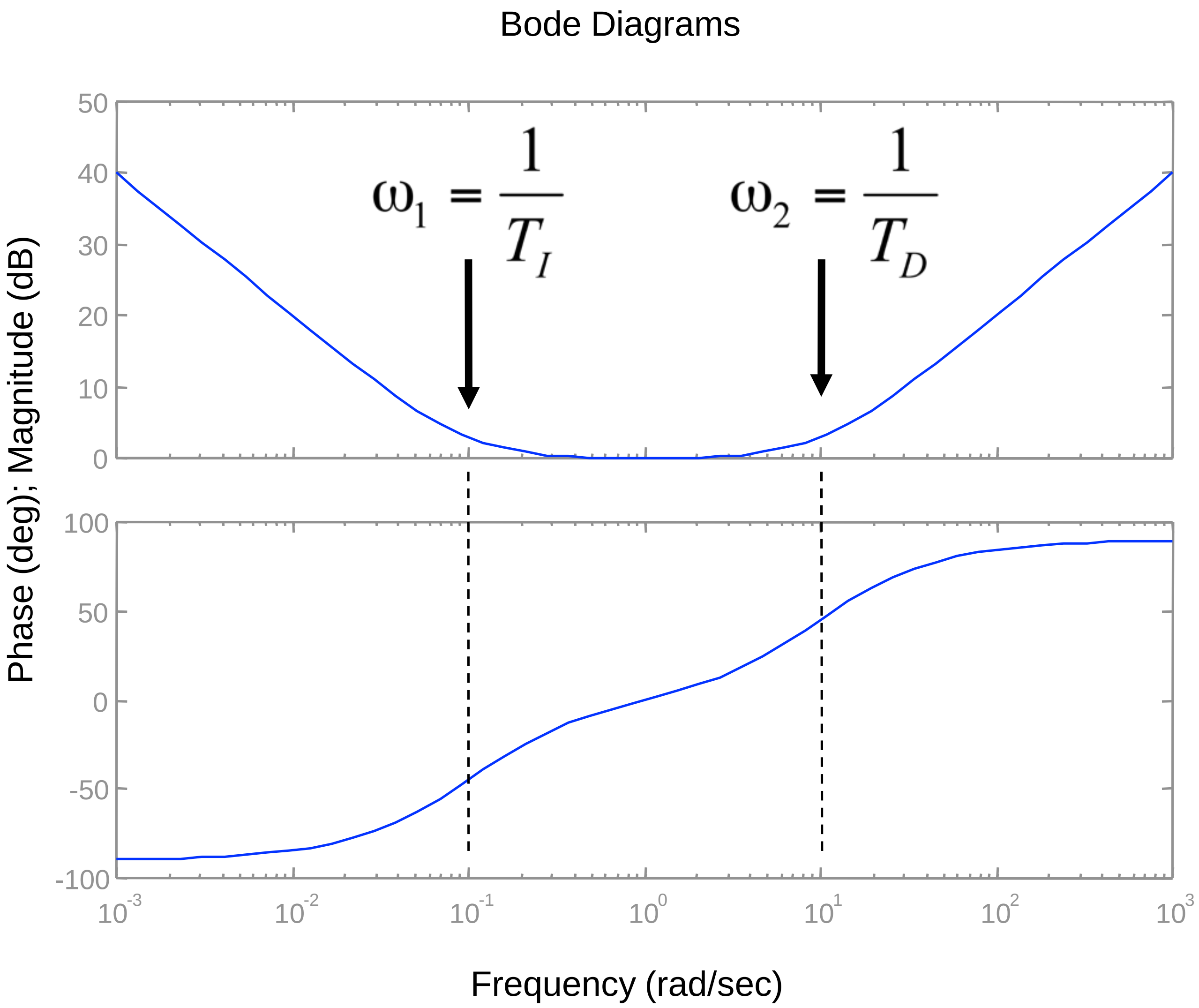
$\omega_z = K_I / K_p$

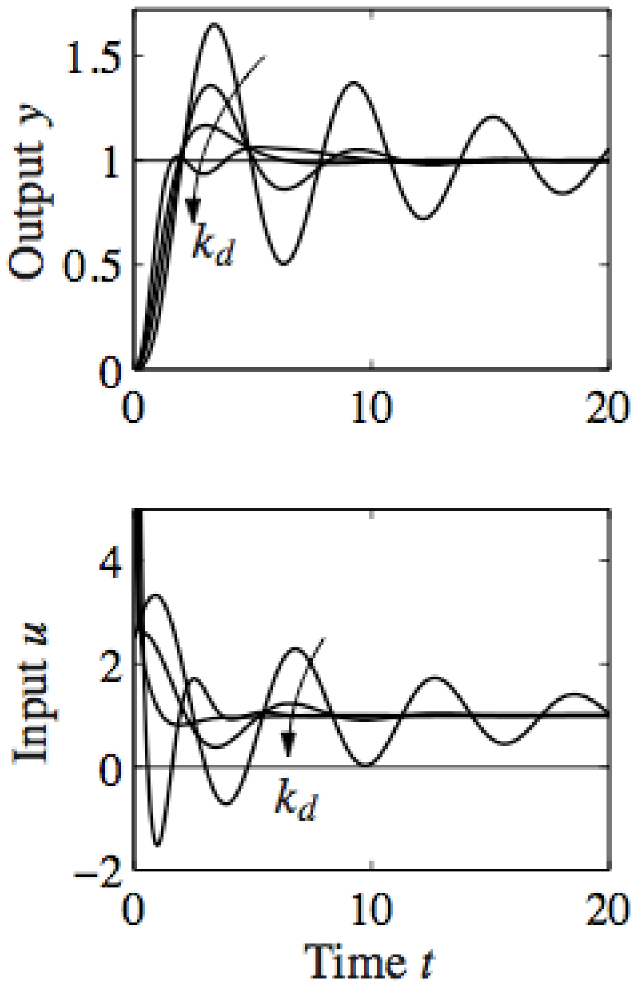


Proportional + Integral + Derivative (PID)

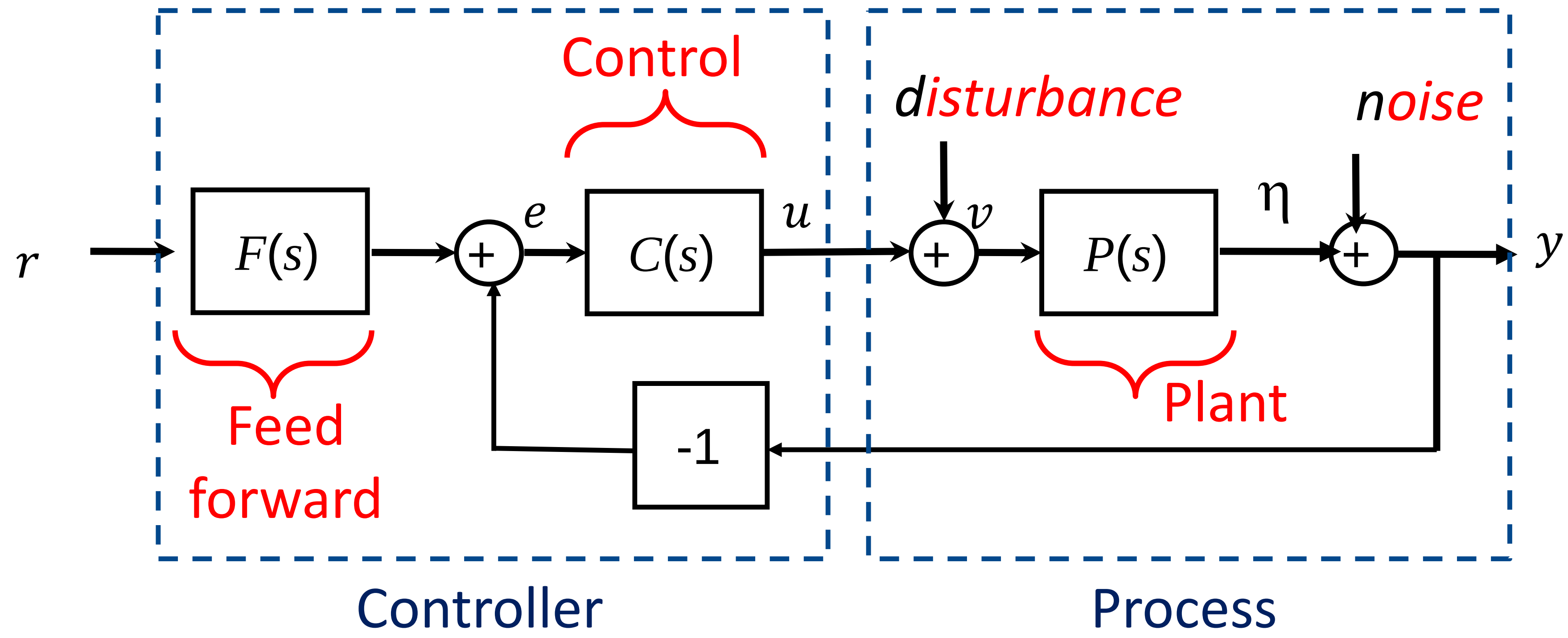


$$\begin{aligned}C(s) &= k_p + k_i \frac{1}{s} + k_d s\\ &= k(1 + \frac{1}{T_i s} + T_d s)\\ &= \frac{kT_d}{T_i} \frac{(s+1/T_i)(s+1/T_d)}{s}\end{aligned}$$





General Loop Transfer Functions



r = reference input
e = error
u = control
v = control + disturbance
η = true output (***what we want to control***!)
y = measured output

System "outputs"

System "inputs"

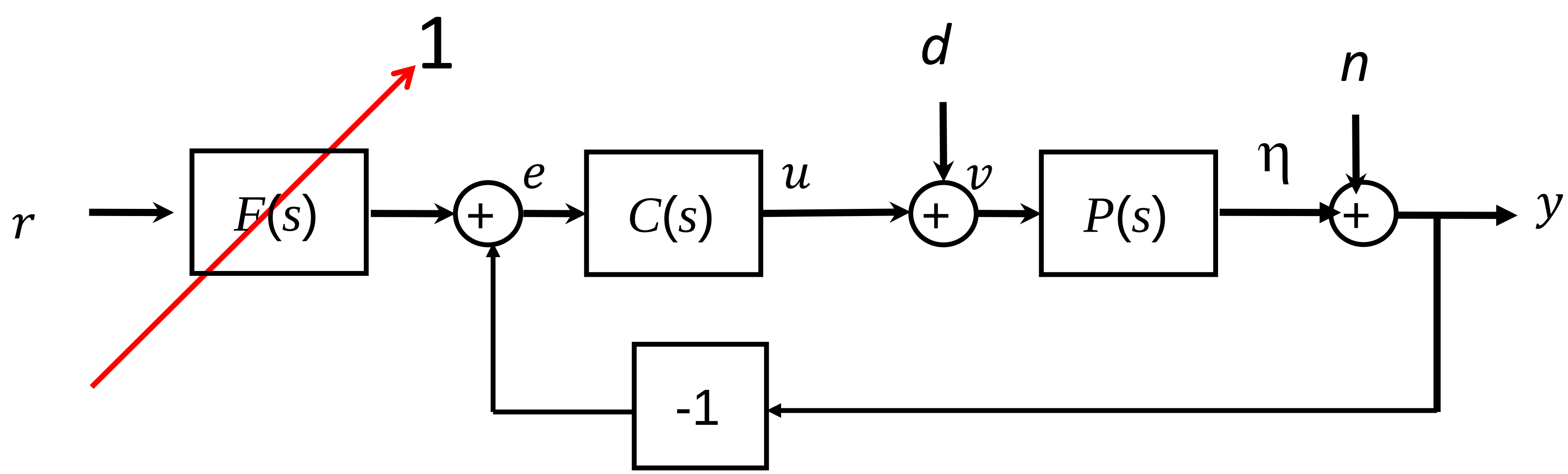
$$\begin{pmatrix} y \\ \eta \\ v \\ u \\ e \end{pmatrix} = \begin{pmatrix} \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{CF}{1+PC} & \frac{1}{1+PC} & \frac{-C}{1+PC} \\ \frac{CF}{1+PC} & \frac{-PC}{1+PC} & \frac{-C}{1+PC} \\ \boxed{\frac{F}{1+PC}} & \frac{-P}{1+PC} & \frac{-1}{1+PC} \end{pmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

"Gang of Seven"

"Gang of Six"

$\text{TF} = \frac{PCF}{1+PC}$	$\text{T} = \frac{PC}{1+PC}$	$\text{PS} = \frac{P}{1+PC}$
$\text{CFS} = \frac{CF}{1+PC}$	$\text{CS} = \frac{C}{1+PC}$	$\text{S} = \frac{1}{1+PC}$
Response of (y, u) to r	Response of u to (d,n)	Response of y to (d,n)

Key Loop Transfer Functions



$F(s)$ = 1: Four unique transfer functions define performance ("Gang of Four")

Sensitivity: Function $$G_{er} = S(s) = \frac{1}{1+L(s)}$$

Complementary Sensitivity Function: $$G_{yr} = T(s) = \frac{L(s)}{1+L(s)}$$

Load Sensitivity Function: $$G_{yd} = PS(s) = \frac{P(s)}{1+L(s)}$$

Noise Sensitivity Function: $$G_{yn} = CS\,(s) = \frac{C(s)}{1+L(s)}$$

$$\boxed{L(s) = P(s)C(s)}$$

"Gang of Four"

(the "sensitivity" functions)

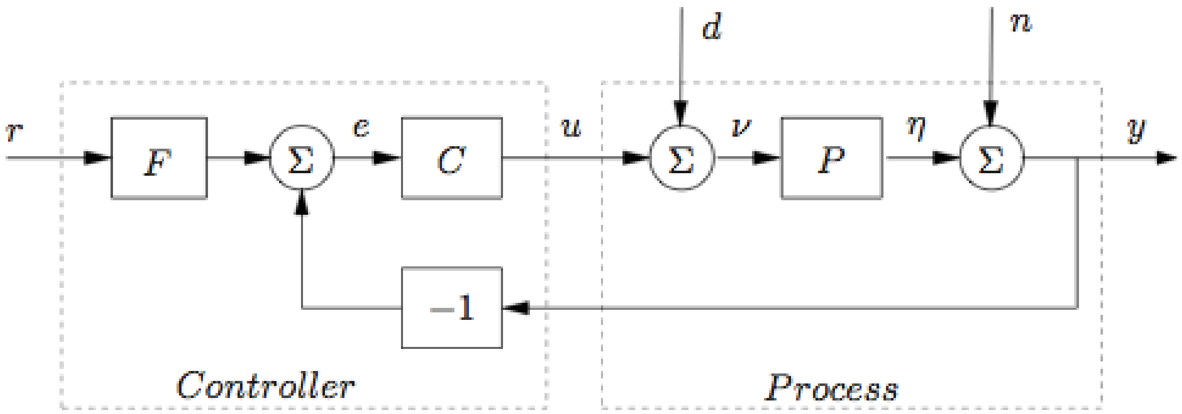
Characterize most performance criteria of interest

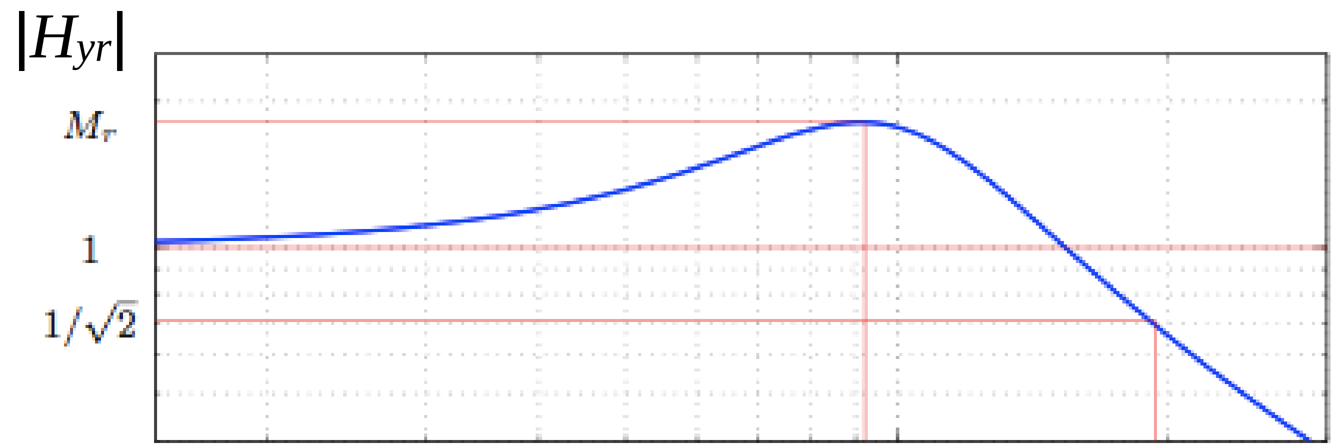
Rough Loop Shaping Design Process

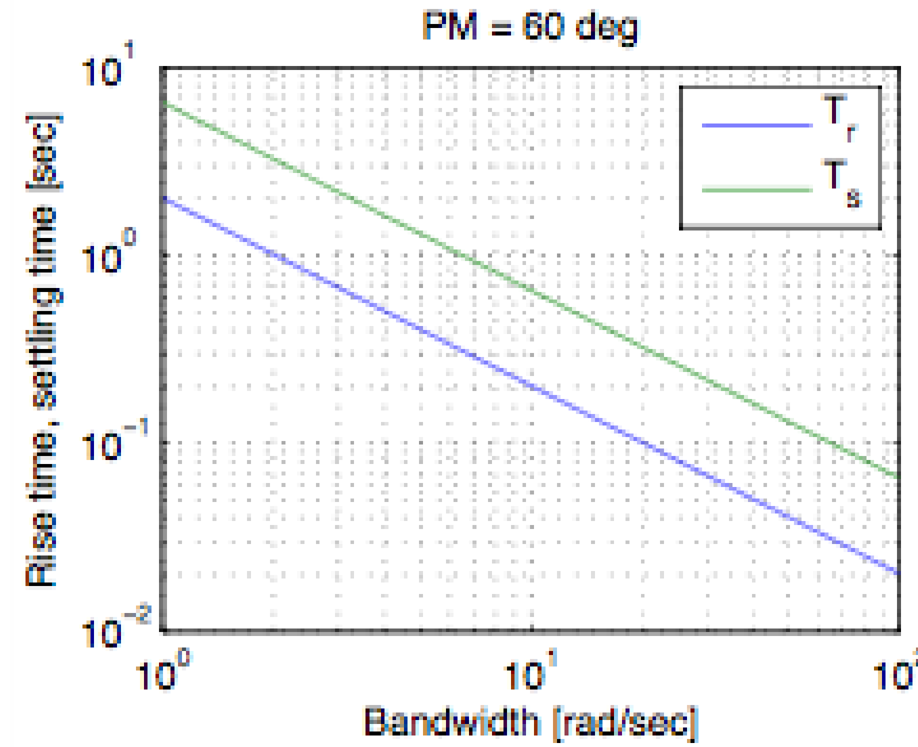
***A Process*: sequence of (nonunique) steps**

1. **Start with plant and performance specifications**
2. **If plant not stable, first stabilize it (e.g., PID)**
3. **Adjust/increase simple gains**
   - Increase proportional gain for tracking error
   - Introduce integral term for steady-state error
   - Will derivative term improve overshoot?
4. **Analyze/adjust for stability and/or phase margin**
   - Adjust gains for margin
   - Introduce *Lead* or *Lag Compensators* to adjust phase margin at crossover and other critical frequencies
   - Consider PID if you haven't already

Summary of Specifications







Key Idea: convert ***closed loop*** *specifications* on

$$G_{yr}(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{L\,(s)}{1 + L(s)}$$

to equivalent specifications on ***loop*** system $L(s)$

- Time domain spec.s can often be converted to frequency domain spec.s

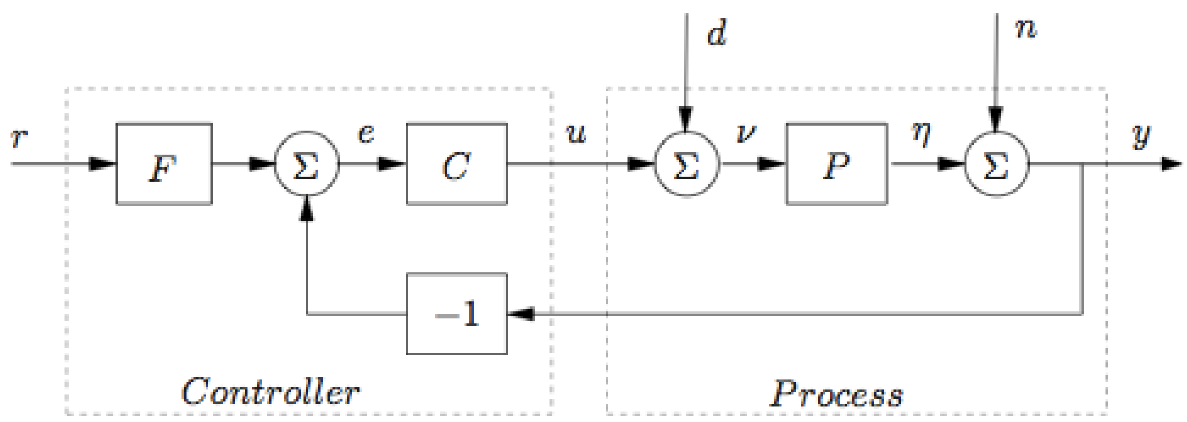
Steady-state tracking error $< X\%$ $\Rightarrow$ $|L(0)| > 1/X$

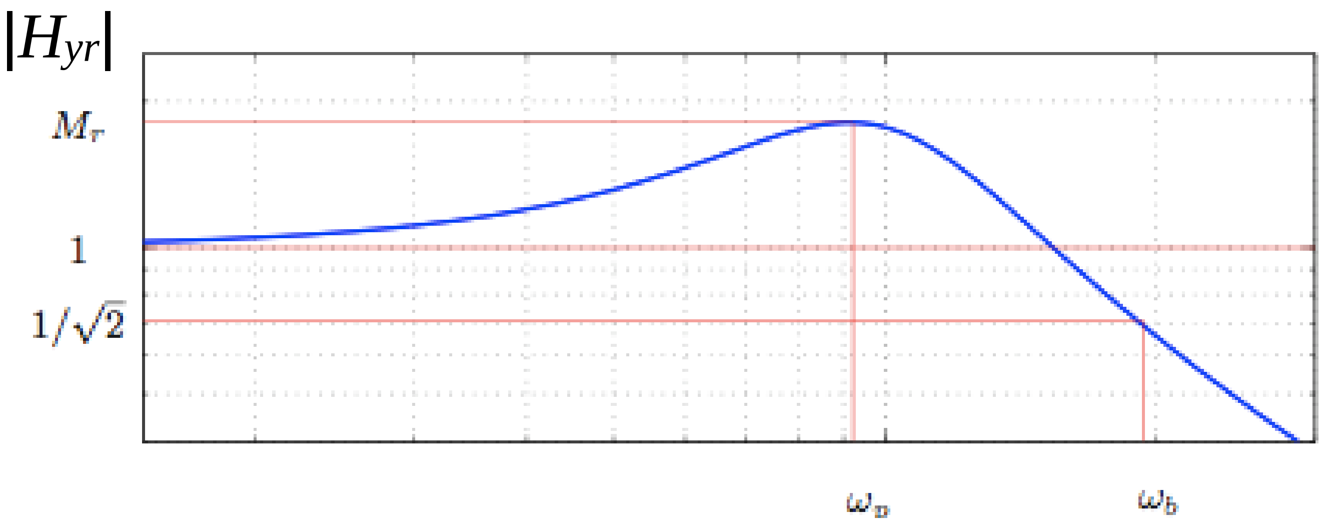
Tracking error $< Y\%$ up to frequency f_t Hz $\Rightarrow$ $|L(i\omega)| > 1/Y$ for $\omega < 2\pi f_t$

Bandwidth of ω_b rad/sec $\Rightarrow$ $|L(i\omega_b)| = 1$

- Usually needed for rise/settling time spec.

Summary of Specifications

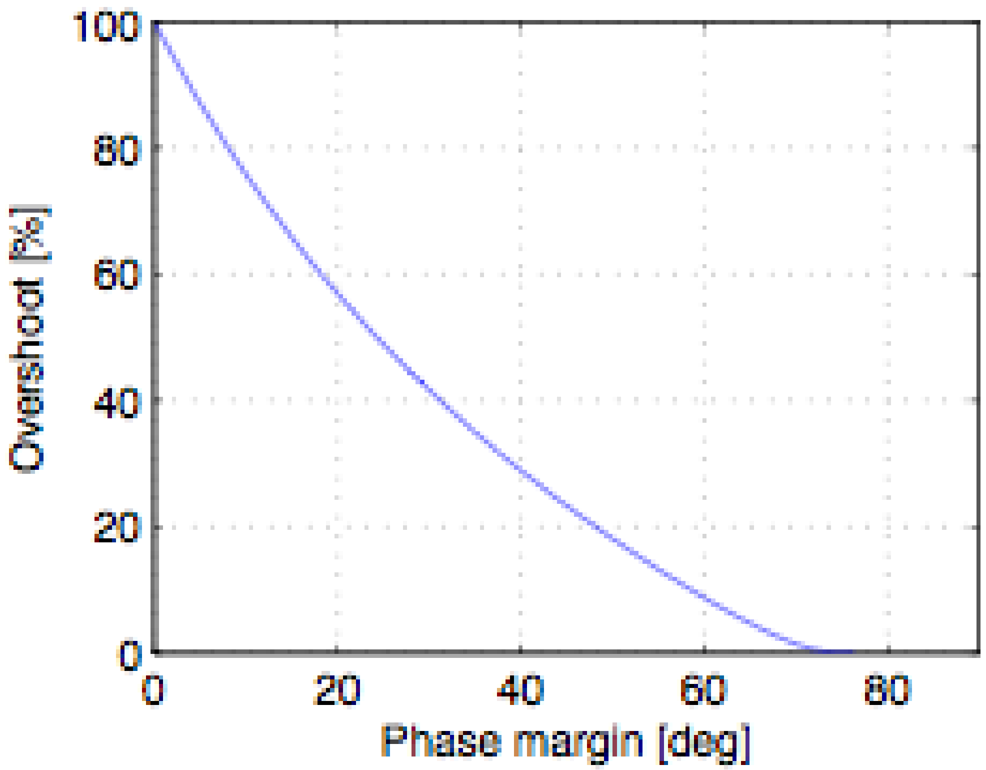




Overshoot $< \ Z\%$ $\Rightarrow$ Phase Margin $> f(Z)$

Phase/Gain margins (Specified Directly)

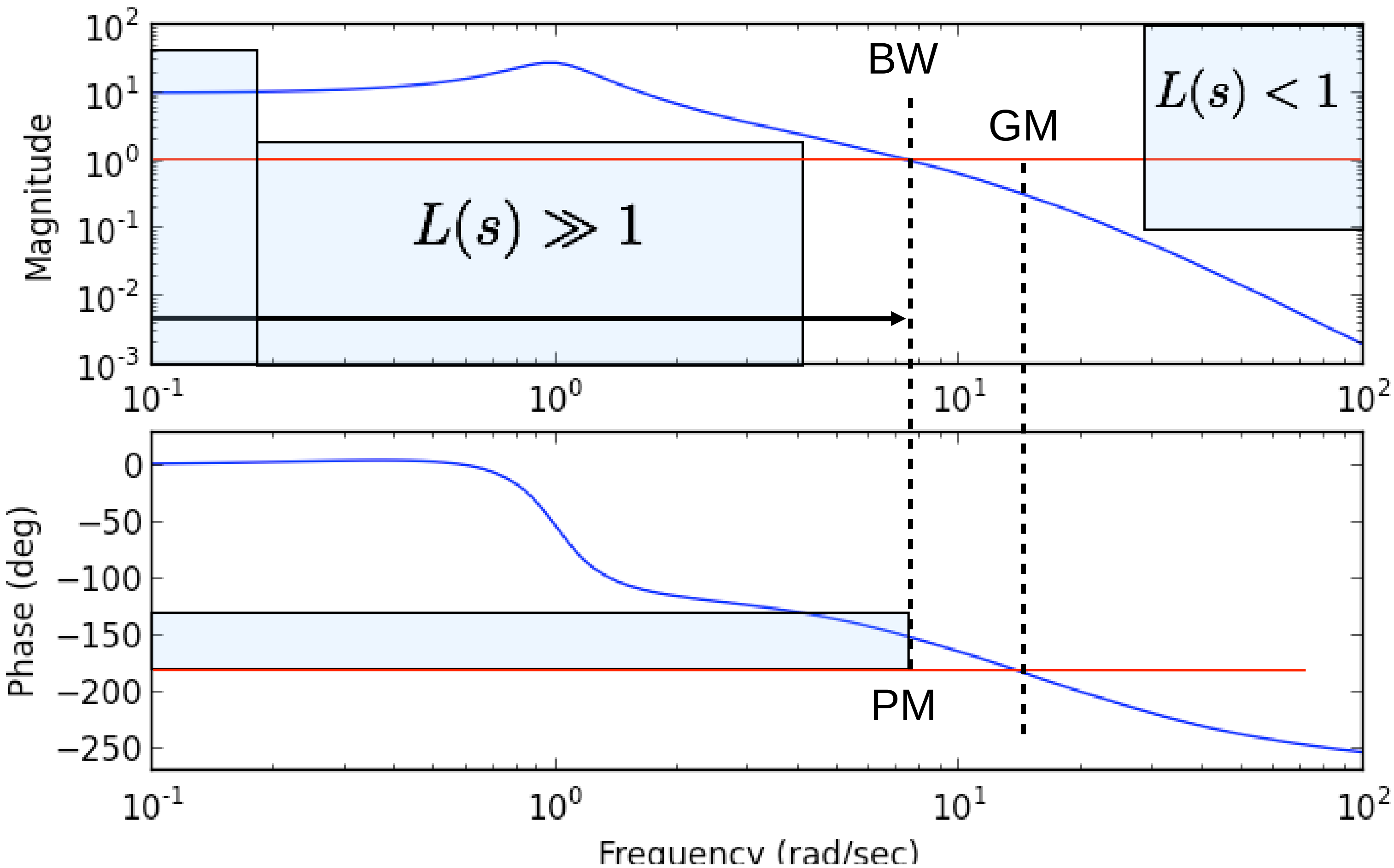
- For robustness
- Typically, at least gain margin of 2 (6 dB)
- Usually, phase margin of 30-60 degrees



Summary: Loop Shaping

Loop Shaping for Stability & Performance

- Steady state error, bandwidth, tracking response
- Specs can be on any input/output response pair

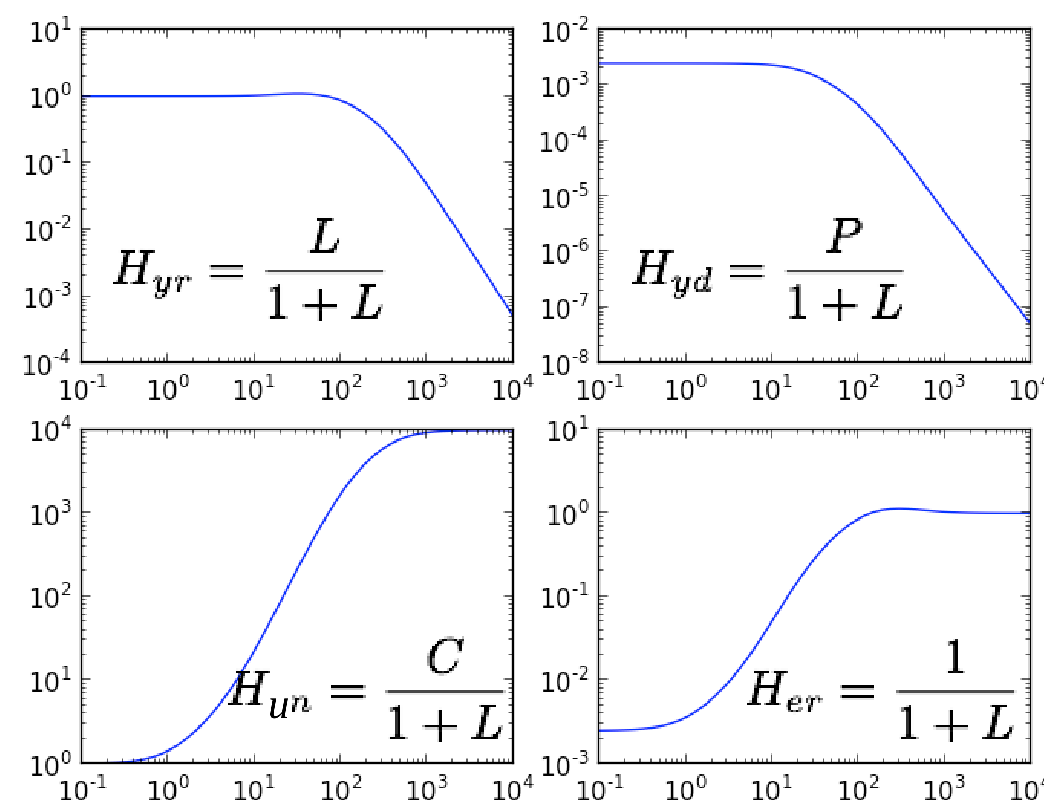


Things to remember (for homework and exams)

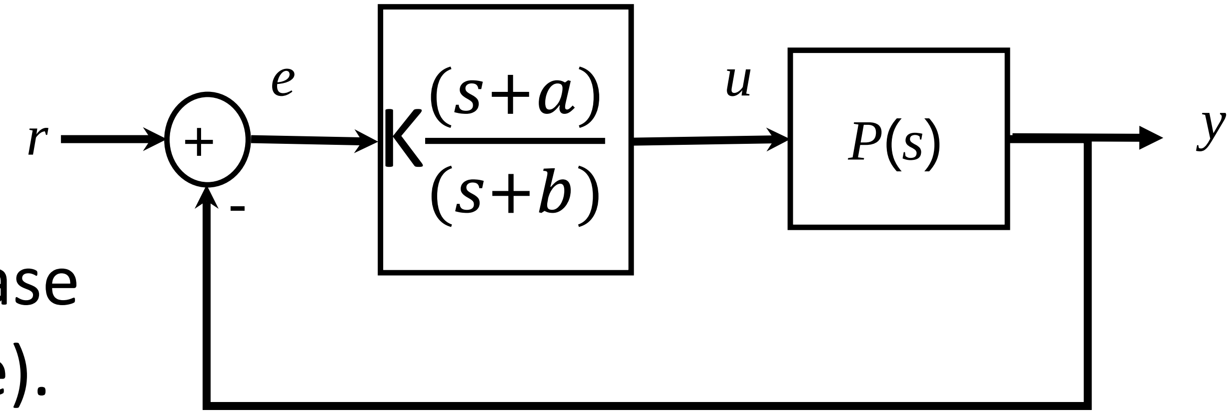
- Always plot Nyquist to verify stability/robustness
- Check gang of 4 to make sure that noise and disturbance responses also look OK

Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI



Lead & Lag Compensators



Lead: K > 0, a < b

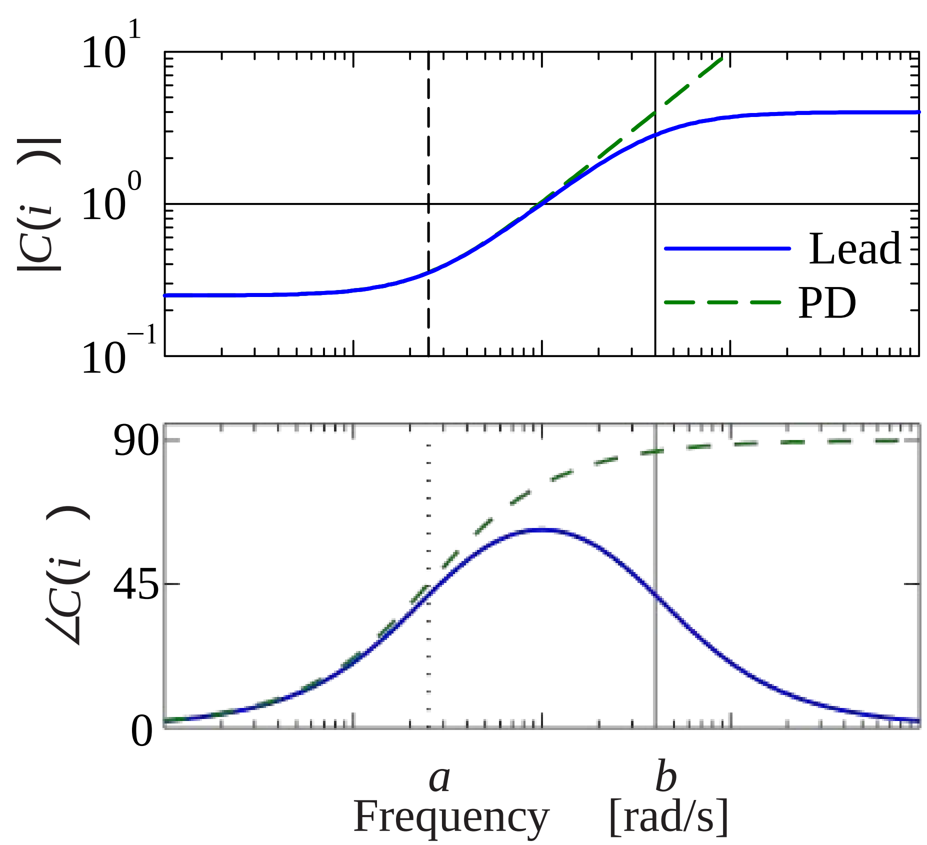
- Add phase near crossover
- Improve gain & phase margins, increase bandwidth (better transient response).

Lag: K > 0, a> b

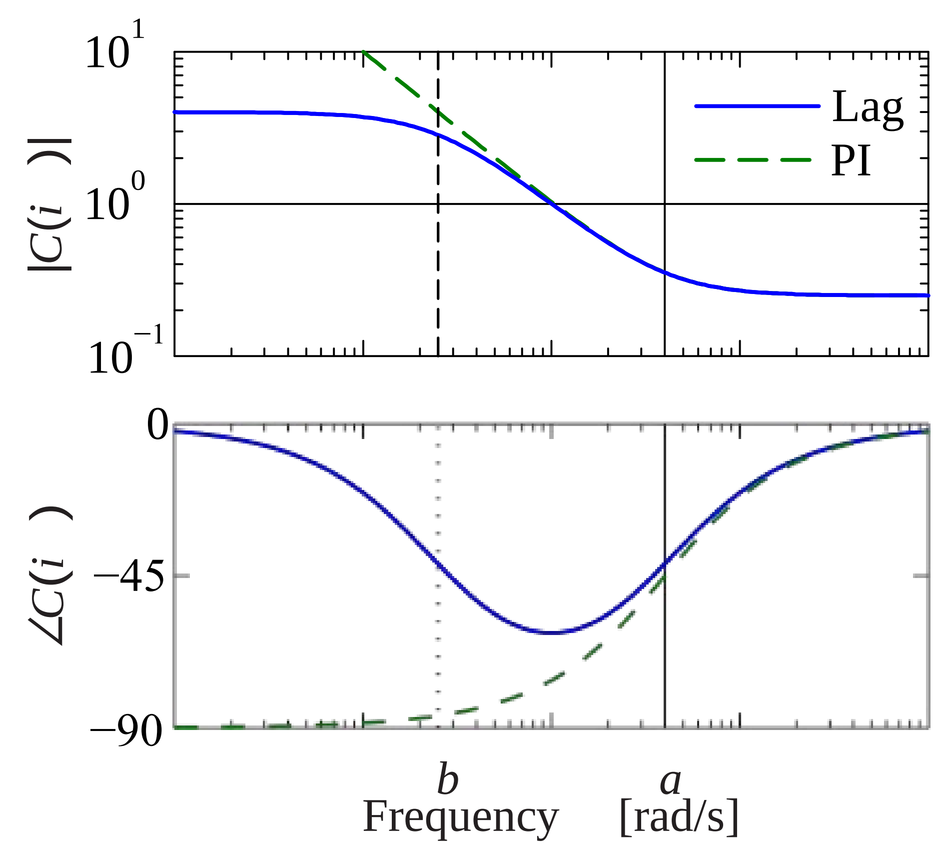
- Add gain in low frequencies
- Improves steady state error

Lead/Lag:

- Better transient and steady state response



(a) Lead compensation, $a < b$



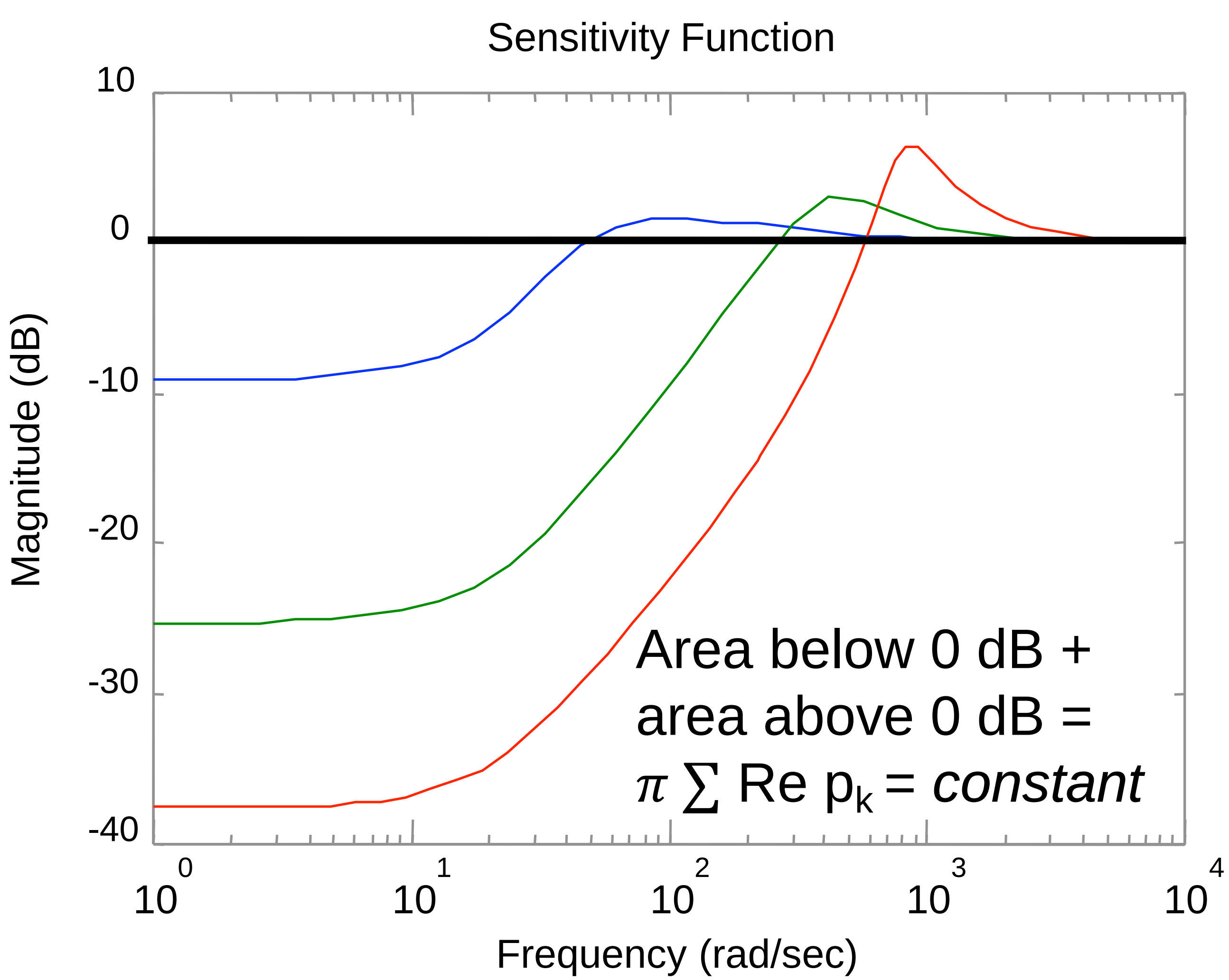
(b) Lag compensation, $b < a$

Bode's Integral Formula and the Waterbed Effect

Bode's integral formula for $S(s) = \frac{1}{1+L(s)} = G_{er} = G_{yn} = G_{vd} = -G_{en}$

- Let p_k be the unstable poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
- **Theorem:** the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1+L(j\omega)|} d\omega = \pi \sum \mathrm{Re}\, p_k$$



Waterbed effect:

- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in ω; Bode plots are logarithmic