Localization Using the KF

Assumption: there are N ***known*** landmarks, with known locations

$$\{x_1^L, x_2^L, \dots, x_N^L\}$$

Use the Kalman filter to update robot's position estimate. The robot's state is x^R. The dynamics are generally of the form:

$$x_{k+1}^R = f\left(x_k^R, u_k\right) + v_k$$

And the measurements are of the form:

$$y_{k+1} = h\left(x^R, x_1^L, \cdots x_N^L\right) + \omega_{k+1}$$

Pseudo-Code: (1) Initialize $\hat{x}_{0|0}^R$ and $P_{0|0}$, $k = 0$

(2) Move to x_{k+1}, estimate $\hat{x}_{k+1|k}$, $P_{k+1|k}$

(3) Measure y_{k+1}, update $\hat{x}_{k+1|k+1}$, $P_{k+1|k+1}$

(4) k=k+1; go to step (2)

Localization Using the KF

(Example)

Turtlebot kinematics:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{R}{2} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 + \dot{\phi}_2 \\ 0 \\ (\dot{\phi}_1 - \dot{\phi}_2)/W \end{bmatrix}$$

We cannot integrate these equations exactly for a discrete time system. Approximate integration:

$$\begin{bmatrix} (x_{k+1} - x_k)/\Delta t \\ (y_{k+1} - y_k)/\Delta t \\ (\theta_{k+1} - \theta_k)/\Delta t \end{bmatrix} = \frac{R}{2} \begin{bmatrix} \cos\theta_k & -\sin\theta_k & 0 \\ \sin\theta_k & \cos\theta_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\phi_{1,k} + \Delta\phi_{2,k} \\ 0 \\ (\Delta\phi_{1,k} + \Delta\phi_{2,k})/W \end{bmatrix}$$

Rearranging:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ \theta_{k+1} \end{bmatrix} = I_{3\times 3} \begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} + \frac{R}{2} \begin{bmatrix} \cos\theta_k & \cos\theta_k \\ \sin\theta_k & \sin\theta_k \\ 1/W & -1/W \end{bmatrix} \begin{bmatrix} \Delta\phi_{1,k} \\ \Delta\phi_{2,k} \end{bmatrix}$$

$$\vec{z}_{k+1} = A_k\, \vec{z}_k + B_k\, u_k$$

Localization Using the KF

(Example)

Range Measurement: range to j^{th} landmark at t_k

$$y_{j,k} = r_{j,k} = \sqrt{\left(x_k^R - x_j^L\right)^2 + (y_k^R - y_j^L)^\wedge 2} = h_j(x^R, x_j^L)$$

Bearing Measurement: bearing to j^{th} landmark at t_k

$$y_{j,k} = \varphi_{j,k} = Atan2\left[\left(y_k^R - y_k^L\right), (x_k^R - x_k^L)\right] - \theta_k$$

Overall Measurement equations:

$$\vec{y}_{k+1} = \begin{bmatrix} r_{1,k+1} \\ r_{2,k+1} \\ \vdots \\ \varphi_{1,k+1} \\ \varphi_{2,k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} h_1(x_{k+1}^R, x_1^L) \\ h_2(x_{k+1}^R, x_2^L) \\ \vdots \\ h_{N+1}(x_{k+1}^R, x_1^L) \\ h_{N+2}(x_{k+1}^R, x_2^L) \\ \vdots \end{bmatrix} + \vec{\omega}_{k+1}$$

Sequential Measurement Update

(when all landmarks not visible)

If the measurements of the different landmarks are *independent*, we can use a *sequential updating method* for the measurement update:

- Init: $\bar{x}_{k+1|k} = \hat{x}_{k+1}$; $\bar{P}_{k+1|k} = P_{k+1|k}$
- Iterate for all viewed landmarks, $i = 1, \dots, N_{view}$
  - $S_{k+1}^{i} = H_{k+1}^{i} \bar{P}_{k+1|k} \left(H_{k+1}^{i}\right)^{T} + R_{k+1}^{i}$
  - $K_{k+1}^{i} = \bar{P}_{k+1|k} \left(H_{k+1}^{i}\right)^{T} \left(S_{k+1}^{i}\right)^{-1}$
  - $\bar{x}_{k+1|k} = \bar{x}_{k+1|k} + K_{k+1}^{i}\left(y_{k+1} - h\left(\bar{x}_{k+1|k}\right)\right)$
  - $\bar{P}_{k+1|k} = \left(I - K_{k+1}^{i} H_{k_1}\right) \bar{P}_{k+1|k}$
- $\hat{x}_{k+1|k+1} = \bar{x}_{k+1|k}$; $\hat{P}_{k+1|k+1} = \bar{P}_{k+1|k}$

Where:

$$S_{k+1|k}^{i} = E\left[\left(y_{k+1} - h\left(\hat{x}_{k+1|k}\right)\right)\left(y_{k+1} - h\left(\hat{x}_{k+1|k}\right)\right)^{\mathrm{T}}\right]$$ =innovation covariance

The EKF

For system: $x_{k+1} = f(x_k, u_k) + \eta_k$ $\quad\quad \eta_k \sim N(0, Q_k)$

$y_{k+1} = h(x_{k+1}) + \omega_k$ $\quad\quad \omega_k \sim N(0, R_k)$

Just like the KF, the EKF has a ***2-step*** structure:

– Dynamic (time) update)

- $\hat{x}_{k+1|k} = f\left(\hat{x}_{k+1|k}, u_k\right);$ $\quad \mathrm{P}_{k+1|k} = A_k \mathrm{P}_{k|k} A_k^T + Q_k$ $\quad A_k = \left.\frac{\partial f(x)}{\partial x}\right|_{\hat{x}_{k+1|k}}$

– Measurement Update

- $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - h(\hat{x}_{k+1|k}))$
- $\mathrm{P}_{k+1|k+1} = \mathrm{P}_{k+1|k} - \mathrm{P}_{k+1|k}\mathrm{H}_{k+1}^{\mathrm{T}}\left(\mathrm{H}_{\mathrm{k+1}}\mathrm{P}_{k+1|k}\mathrm{H}_{\mathrm{k+1}}^{\mathrm{T}} + \mathrm{R}_{k+1}\right)^{-1}\mathrm{H}_{k+1}\mathrm{P}_{k+1|k}$

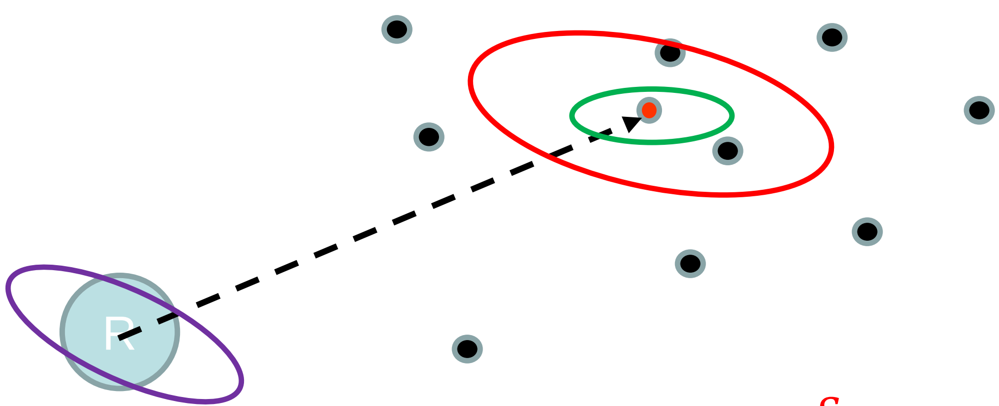
$$= (I - K_{k+1}H_{k+1})\mathrm{P}_{k+1|k}$$

Where: $K_{k+1} = \mathrm{P}_{k+1|k} H_k^T\left(\mathrm{H}_{\mathrm{k+1}}\mathrm{P}_{k+1|k}\mathrm{H}_{\mathrm{k+1}}^{\mathrm{T}} + \mathrm{R}_{k+1}\right)^{-1};$ $\quad H_k = \left.\frac{\partial h(x)}{\partial x}\right|_{\hat{x}_{k+1|k+1}}$

Data Association (1)

In practice we may not exactly know the "identiy" of the landmark. Hence, we must solve the *data association* problem, which tries to assign a label or index to each landmark.

- Innovation: $\nu_{ij} = y^{i}_{k+1} - h^{j}(x_{k+1|k})$
  - measurement error if j^{th}landmark generates i^{th} measurement
- Gate: Eliminate all possible measurement-landmark pairings



R_{k+1} (measure noise)

$P_{k+1|k}$ (Pose Uncertainty)

$S_{k+1|k} = H_{k+1} P_{k+1|k} H_{k+1} + R_{k+1}$

= predicted measurement error

Data Association (2)

In practice we may not exactly know the "identity" of the landmark. Hence, we must solve the *data association* problem, which tries to assign a label or index to each landmark.

- Innovation: $\nu_{ij} = y^{i}_{k+1} - h^{j}\left(x_{k+1|k}\right)$
  - measurement error if j^{th}landmark generates i^{th} measurement
- S_{ij}=innovation uncertainty$\vec{x}^{m}$

Optimal Data Association (or pairing of measurement with landmark index):

- Associate measurement i to the landmark j which minimizes

$$\nu_{ij}^{T} S_{ij}^{-1} \nu_{ij}$$

over all j that survived the "gate"

EKF SLAM

Map: a set of landmarks, which can be modeled as a set of N landmarks and their positions:

$$\vec{x}^m = \left[x_1^L, x_2^L, \dots, x_N^L\right]$$

- **Option 1:** Positions not known in advance, but N is known
- **Option 2:** Even N is not known.

Use Localization EKF with modification that state is: $[\vec{x}^r, \vec{x}^m]^T$

- **Dynamics:** $\begin{bmatrix} x_{k+1}^r \\ x_{k+1}^m \end{bmatrix} = \begin{bmatrix} f(x_k^r, u_k) \\ x_k^m \end{bmatrix} + \begin{bmatrix} \vec{\eta}_k \\ \vec{0} \end{bmatrix}$
- **Measurement:** $y_{k+1}^i = h\left(\vec{x}_{k+1}^r, \vec{x}^i\right) + \omega_{k+1}^i$ (for i^{th} landmark)

Key issues:

- How to properly "initialize" the filter state and covariance when a new landmark is first seen?
- How to properly "extend" the state when new landmarks arise?

EKF SLAM/Tracking

Map+ Targets: a set of *static* landmarks, and a set of moving *targets:*

$$\vec{x}^m = \left[x_1^L, x_2^L, \dots, x_N^L\right]; \quad \vec{x}_k^t = [x_{1,k}^t, x_{2,k}^t, \dots, x_{N_t,k}^t]$$

Use EKF with state: $[\vec{x}^r,\ \vec{x}^t,\ \vec{x}^m]^T$

- **Dynamics:** $\begin{bmatrix} x_{k+1}^r \\ x_{k+1}^t \\ x_{k+1}^m \end{bmatrix} = \begin{bmatrix} f^r(x_k^r, u_k) \\ f^t(x_k^r, x_k^t) \\ x_{k+1}^m \end{bmatrix} + \begin{bmatrix} \eta_k^r \\ \eta_k^t \\ 0 \end{bmatrix}$
- **Measurement:** $y_{k+1} = h\left(\vec{x}_{k+1}^r, \vec{x}_{k+1}^t, \vec{x}_{k+1}^m\right) + \omega_{k+1}$

Key issues:

- Dynamics of target?

  $x_{k+1}^t = \begin{bmatrix} I & \Delta t\ I \\ 0 & I \end{bmatrix} x_k^t + \begin{bmatrix} 0 \\ \eta_k \end{bmatrix}$ (η_k captures possible accelerations)

- Confusion of Targets (use multiple hypothesis tracking for data association)

EKF SLAM: Landmark Initialization

First time you see a landmark: need to

- Initialize its position (after dynamic update)

$$\begin{bmatrix} x_k^{L_i} \\ y_k^{L_i} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k+1|k}^r \\ \hat{y}_{k+1|k}^r \end{bmatrix} + \begin{bmatrix} r_{k+1}^i \cos(\phi_{k+1}^i + \hat{\theta}_{k+1}^t) \\ r_{k+1}^i \sin(\phi_{k+1}^i + \hat{\theta}_{k+1}^t) \end{bmatrix}$$

- Initialize landmark position uncertainty. If N known, let F^i be:

$$H_{k+1}^i = \begin{bmatrix} \frac{\partial h^i}{\partial x^r} & \frac{\partial h^i}{\partial x^{L_i}} \end{bmatrix} \begin{bmatrix} I & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & I & \ldots & 0 \end{bmatrix}$$

- Assuming independent landmark measurements (not quite true…), sequential update:
  - $K_{k+1}^i = \bar{P}_{k+1|k} \left(H_{k+1}^i\right)^T \left(H_{k+1}^i \bar{P}_{k+1|k} \left(H_{k+1}^i\right)^T + R_{k+1}\right)^{-1}$
  - $\bar{x}_{k+1|k} = \bar{x}_{k+1|k} + K_{k+1}^i (y_{k+1} - h(\bar{x}_{k+1|k}))$
  - $\bar{P}_{k+1|l} = \left(I - K_{k+1}^i H_{k_1}\right) \bar{P}_{k+1|k}$
- $\hat{x}_{k+1|k+1} = \bar{x}_{k+1|k}$; $\hat{P}_{k+1|k+1} = \bar{P}_{k+1|k}$

Gmapping: how is it related?

Notation: $x_{1:T} = \{x_1, x_2, \ldots, x_T\}$

Localization: $p(x_{k+1}^r | \mathrm{x}_{1:\mathrm{k}}^{\mathrm{r}}, \mathrm{y}_{1:\mathrm{k}+1})$

- In KF/EKF, $\hat{x}_{k+1}^r$, $P_{k+1|k+1}$ are *sufficient statistics*

EKF SLAM: $p\left(x_{k+1}^r, \mathrm{x}_{\mathrm{k}+1}^{\mathrm{m}} \middle| \mathrm{x}_{1:k}^{\mathrm{r}}, \mathrm{x}_{\mathrm{k}}^{\mathrm{m}}, \mathrm{y}_{1:k+1}\right)$

Occupancy Grid: "map" is a grid of "cells": $\{x_{i,j}^m\}$

- $x_{i,j}^m = 1$ if cell (i,j) is empty; $x_{i,j}^m = 0$ if cell (i,j) is occupied
- $p\left(x_{k+1}^r, \{\mathrm{x}_{i,j}^{\mathrm{m}}\}_{k+1} \middle| \mathrm{x}_{1:k}^{\mathrm{r}}, \{\mathrm{x}_{\mathrm{i,j}}^{\mathrm{m}}\}_k, \mathrm{y}_{1:k+1}\right)$ (estimates cell occupancy probability)

Gmapping:

- Uses a *Rao-Blackwellized* particle filter for estimator
- Actually computes $p\left(x_{1:T}^r, \{\mathrm{x}_{i,j}^{\mathrm{m}}\} \middle| \mathrm{x}_{1:k}^{\mathrm{r}}, \mathrm{x}_{\mathrm{k}}^{\mathrm{m}}, \mathrm{y}_{1:k+1}\right)$