CDS 101/110: Lecture 6.2
Transfer Functions

November 2, 2016

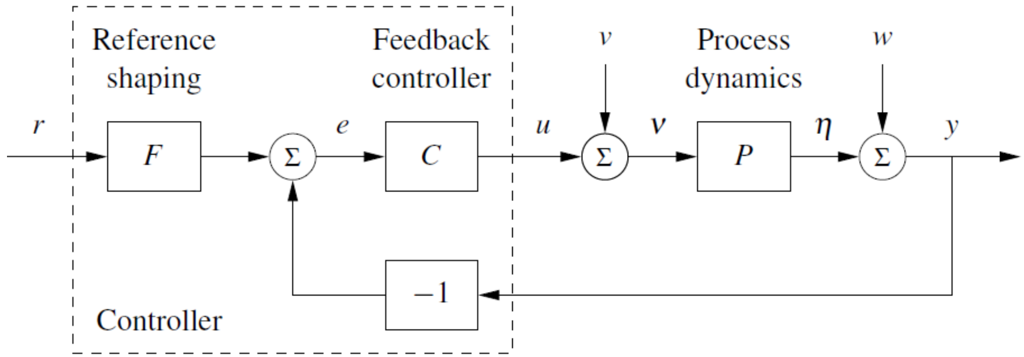
Goals:

- Continued study of Transfer functions
- Review Laplace Transform
- “Block Diagram Algebra”
- Bode Plot Intro

Reading:

- Åström and Murray, Feedback Systems-2e, Chapter 9

Transfer Functions



Exponential response of a linear state space system

$$y(t) = Ce^{At}\big(x(0) - (sI - A)^{-1}B\big) + \big(C(sI - A)^{-1}B + D\big)e^{st}$$

transient steady state

Transfer function

- Steady state response is proportional to exponential input => look at input/output ratio y(s)/u(s)
- $G(s) = C(sI - A)^{-1}B + D$ is the transfer function between input and output

Frequency response (system response to pure sinusoidal input)

$$u(t) = M sin\omega t = \frac{M}{2i}\left(e^{i\omega t} - \mathrm{e}^{-\mathrm{i}\omega \mathrm{t}}\right)$$

$$y_{ss}(t) = \frac{M}{2i}\big(G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\big) = M\,|G(i\omega)|\;\sin(\omega t + \arg G(i\omega))$$

gain phase

Laplace Transform Review

Definition: given a "suitable" function $y(t)$ the Laplace Transform of y is:

$$\mathcal{L}\{y(t)\} = \int_0^\infty y(\tau)e^{-s\tau}\,d\tau$$

Defined on entire complex plane

Properties:

- **Linearity:** $\mathcal{L}\{\alpha\, f(t) + \beta\, g(t)\} = \alpha\,\mathcal{L}\{\,f(t)\} + \beta\,\mathcal{L}\{g(t)\}$, for $\alpha, \beta \epsilon \mathbb{R}$
- **Derivative:** $\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = s\mathcal{L}\{x(t)\} - x(0)$
  - More generally, $\mathcal{L}\left\{\frac{d^n}{dt^n}x(t)\right\} = s^n\mathcal{L}\{x(t)\} - \sum_{k=1}^{n} s^{k-1}\; x^{n-k}(0)$
  - **Proof:**

$$\int_0^\infty \frac{d}{dt}(x(t)e^{-st})\,dt = x(t)e^{-st}\Big|_0^\infty = \int_0^\infty \frac{dx}{dt}e^{-st}dt - s\int_0^\infty x(t)e^{-st}dt$$

$$-x(0) = \mathcal{L}\left\{\frac{dx}{dt}\right\} - s\mathcal{L}\{x(t)\}$$

Laplace Transform Review

- **Integral:** $\mathcal{L}\left\{\int_0^t x(t)\,dt\right\} = \frac{1}{s}\mathcal{L}\{x(t)\}$

- **Convolution:** $\mathcal{L}\{h(t) * g(t)\} = H(s)G(s)$

  Proof: $\mathcal{L}\{h(t) * g(t)\} = \int_0^{\infty}\big(h(t) * g(t)\big)e^{-st}dt$

  If h(t) is *causal* (h(t) =0 for t<0), then $h(t) * g(t) = \int_0^t h(t-\tau)g(\tau)d\tau$

  $$\mathcal{L}\{h(t) * g(t)\} = \int_0^{\infty}\int_0^{t} h(t-\tau)\,g(\tau)e^{-st}d\tau\,dt = \int_0^{\infty}\int_{\tau}^{\infty} h(t-\tau)\,g(\tau)e^{-st}dtd\tau$$

  $$= \int_0^{\infty}\int_0^{\infty} h(\mu)\,g(\tau)\,e^{-s(\mu+\tau)}d\mu\,d\tau = \int_0^{\infty} h(\mu)e^{-s\mu}d\mu\int_0^{\infty} g(\tau)e^{-s\tau}d\tau = H(s)G(s)$$

- **Exponential:**
  - $\mathcal{L}\{e^{-at}\} = \int_0^{\infty} e^{-(s+a)t}\,dt = \frac{-1}{s+a}e^{-(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$
  - $\mathcal{L}\left\{e^{-At}\right\} = (sI - A)^{-1} = \frac{adj(sI-A)}{\det(sI-A)}$

Laplace Transform Review

- **Transfer Function:**

$$\dot{x} = Ax + Bu \quad \Rightarrow_{\mathcal{L}} \quad sX(s) - x(0) = AX(s) + BU(S)$$

$$y = Cx + Du \quad \Rightarrow_{\mathcal{L}} \quad Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}B\,U(s) + (sI - A)^{-1}x(0)$$

$$Y(s) = \left(C(sI - A)^{-1}B + D\right)U(s) + C(sI - A)^{-1}x(0)$$

When $x(0) = 0$, $\Rightarrow$ $\frac{Y(s)}{U(s)} = G(s) = C(sI - A)^{-1}B + D$

- **Impulse Response:** $\mathrm{y(t)} = \int_0^\infty h(t-\tau)u(\tau)d\tau$

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}\,y(t)dt = \int_0^\infty e^{-st}\int_{0\infty}^\infty h(t-\tau)u(\tau)d\tau\,dt$$

$$= \int_0^\infty \int_0^t e^{-s(t-\tau)}e^{-s\tau}h(t-\tau)u(\tau)d\tau dt = \int_0^\infty e^{-s\tau}\,u(\tau)d\tau \int_0^\infty e^{-st}h(t)dt$$

$$= H(s)U(s)$$

Therefore, $G(s) = \mathcal{L}\{h(t)\}$=Laplace Transform of system Impulse Response

Laplace Transform Review

Constant Coefficient O.D.E.: Laplace Transform (assuming zero initial conditions)

$$\frac{d^n}{dt^n}y(t) + a_1\frac{d^{n-1}}{dt^{n-1}}y(t) + \cdots + a_n y(t) = b_1\frac{d^{n-1}}{dt^{n-1}}u(t) + \cdots + b_n u(t) \quad (*)$$

$\mathcal{L}\{\cdot\}$

$$\left(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\right)Y(s) = \left(b_1 s^{n-1} + \cdots + b_n\right)U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\left(b_1 s^{n-1} + \cdots + b_n\right)}{\left(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\right)} = \frac{n(s)}{d(s)}$$

- Roots of $d(s)$ are called the *poles* of transfer function $G(s)$
  - If p is a system pole, then $y = e^{pt}$ is a solution to (*) with $u(t) = 0$
  - Poles are strictly defined by matrix A..
- Roots of $n(s)$ are called the *zeros* of G(s)
  - If s is a pole of $G(s)$, then $G(s)e^{st}$ is an output if $d(s) \neq 0$.
  - Out put is zero at s if $n(s) = 0$.

Poles and Zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

- Roots of *d*(*s*) are called *poles* of *G*(*s*)
- Roots of *n*(*s*) are called *zeros* of *G*(*s*)

Poles of G(s) determine the stability of the (closed loop) system

- Denominator of transfer function = characteristic polynomial of state space system
- Provides easy method for computing stability of systems
- Right half plane (RHP) poles (Re > 0) correspond to unstable systems

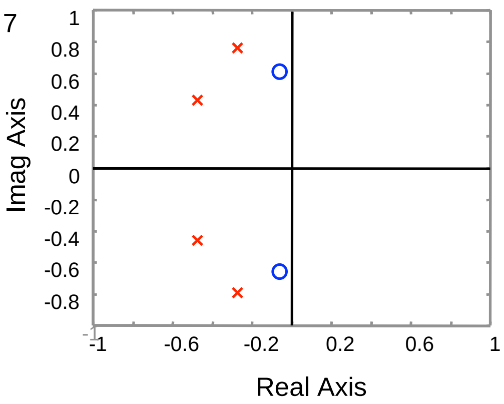
Zeros of G(s) related to frequency ranges with limited transmission

- A pure imaginary zero at s = iω blocks any output at that frequency (G(iω) = 0)
- Zeros provide limits on performance, especially RHP zeros

MATLAB: pole(G), zero (G), pzmap(G)

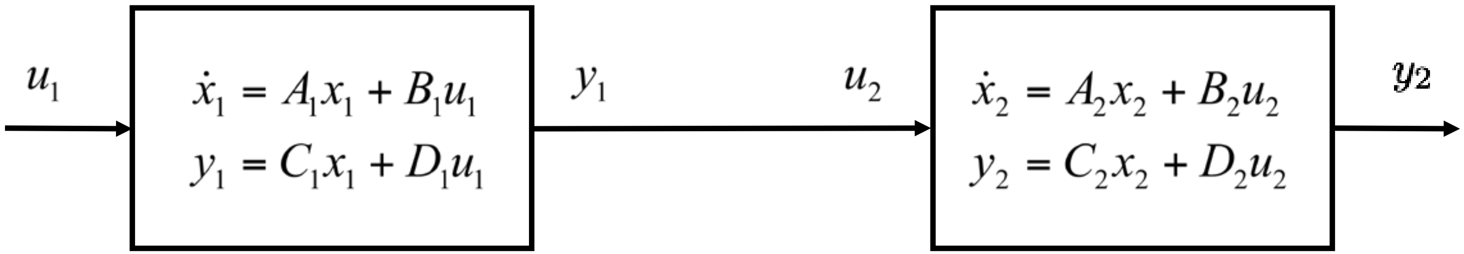
$$G(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

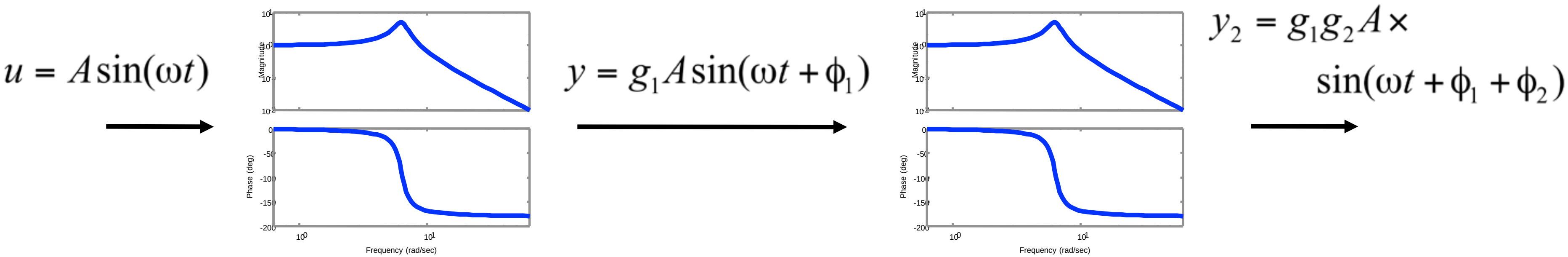
pzmap(G)



Series Interconnections

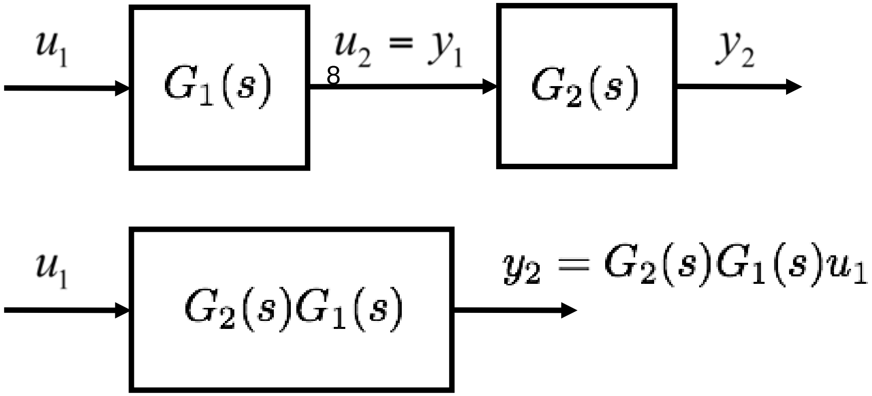
Q: what happens when we connect two systems together in series?



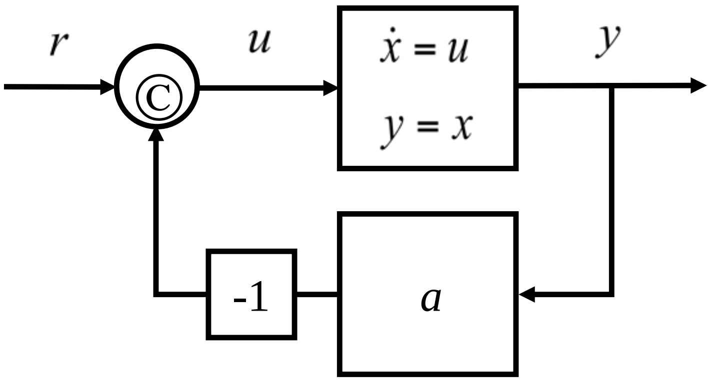


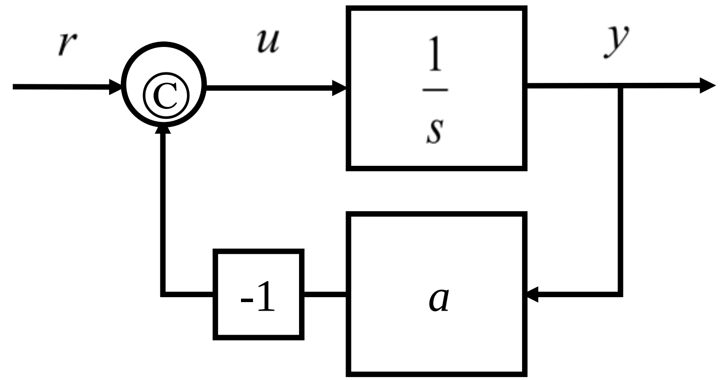
A: Transfer functions multiply

- Gains multiply, phases add
- Generally: transfer functions well formulated for frequency domain interconnections
  - Convolution → multiplication
- MATLAB/python: G = series(G1, G2)



Feedback Interconnection





State space derivation

$$\dot{x} = u = r - ay = -ax + r$$
$$y = x$$

Frequency response $r = A\sin(\omega t)$

$$y = \left|\frac{1}{\sqrt{a^2+\omega^2}}\right| \sin\left(\omega t - \tan^{-1}\left(\frac{\omega}{a}\right)\right)$$

Transfer function derivation

$$y = \frac{u}{s} = \frac{r - ay}{s}$$
$$y = \frac{r}{s+a} = G(s)r$$

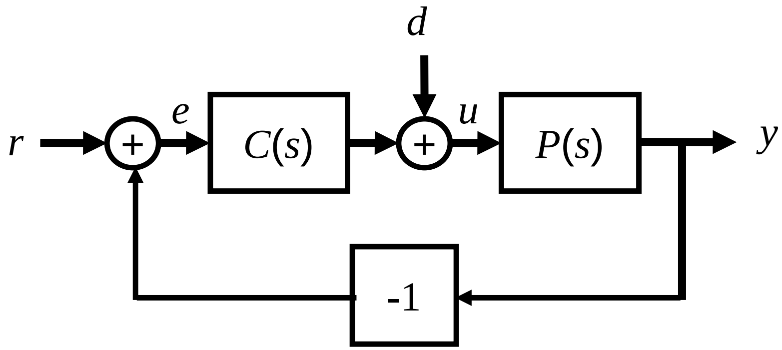
Frequency response

$$y = |G(i\omega)|\sin(\omega t + \angle G(i\omega))$$

- MATLAB: G = feedback(sys, a) $\leftarrow$ works for either state space or transfer functions

Block Diagram Algebra

Basic idea: treat transfer functions as multiplication, write down equations



$$y = P(s)u$$
$$u = d + C(s)e$$
$$e = r - y$$

Manipulate equations to compute desired signals

$$\begin{aligned} e &= r - y \\ &= r - P(s)u \\ &= r - P(s)(d + C(s)e) \end{aligned}$$

$$(1 + P(s)C(s))e = r - P(s)d$$

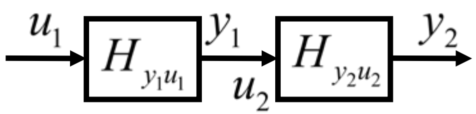
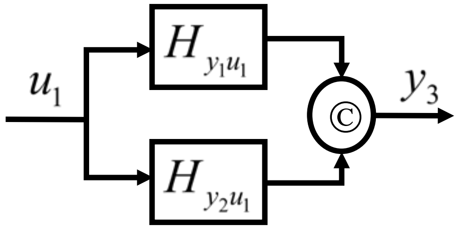
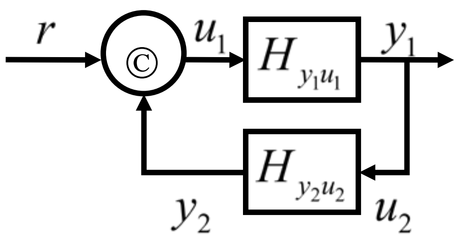
$$e = \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d$$

Note: linearity gives super-position of terms

Algebra works because we are working in frequency domain

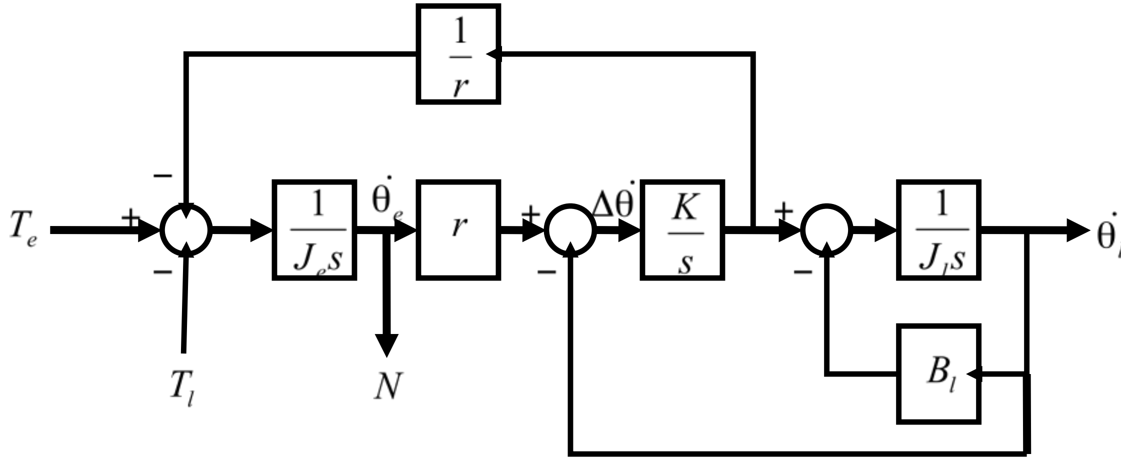
- Time domain (ODE) representations are not as easy to work with
- Formally, all of this works because of Laplace transforms

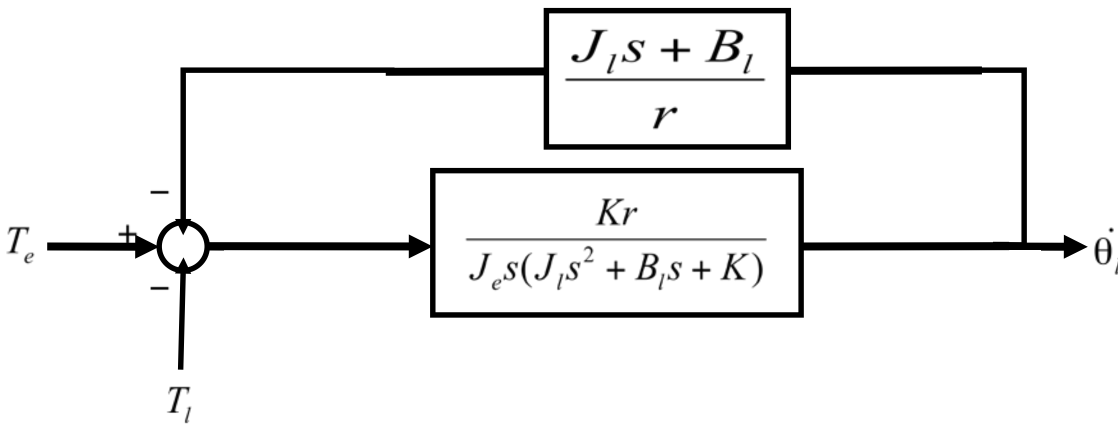
Block Diagram Algebra

Type	Diagram	Transfer function
Series		$H_{y_2u_1} = H_{y_2u_2} H_{y_1u_1} = \dfrac{n_1 n_2}{d_1 d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \dfrac{n_1 d_2 + n_2 d_1}{d_1 d_2}$
Feedback		$H_{y_1r} = \dfrac{H_{y_1u_1}}{1 + H_{y_1u_1} H_{y_2u_2}} = \dfrac{n_1 d_2}{n_1 n_2 + d_1 d_2}$

- These are the basic manipulations needed; some others are possible
- Formally, could work all of this out using the original ODEs (⇒ nothing really new)

Example: Engine Control of a GM Astro





$$H_{\theta_l T_e}(s) = \frac{Kr}{J_e J_l s^3 + J_e B_l s^2 + (J_e K + K J_l)s + K B_l}$$

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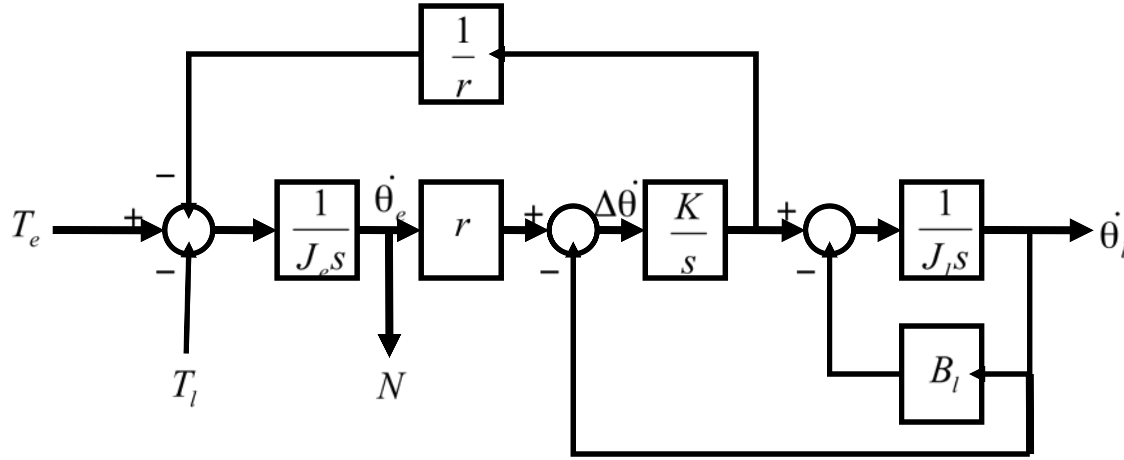

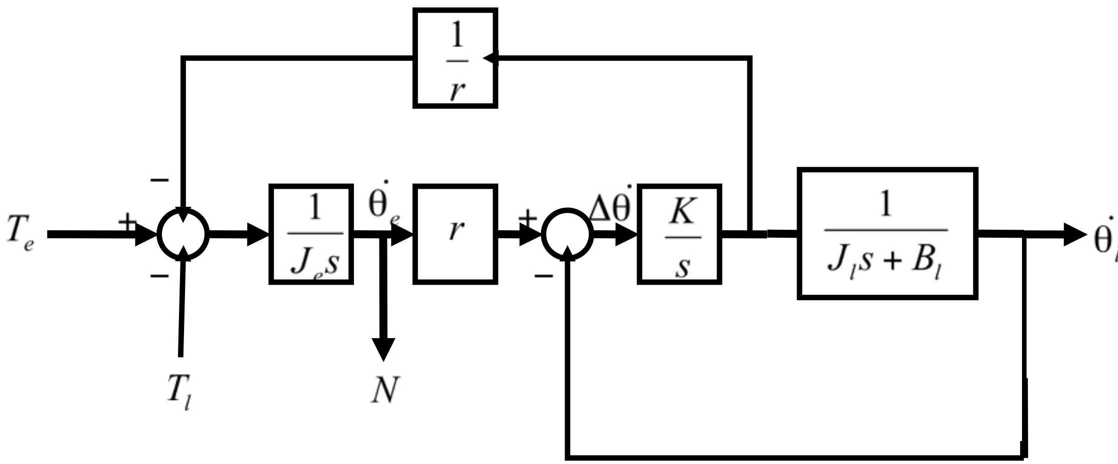

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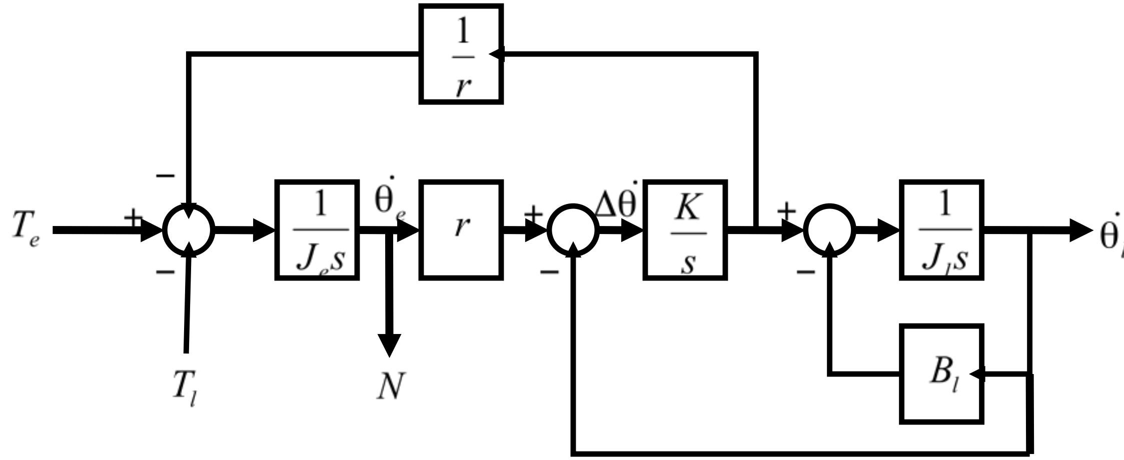

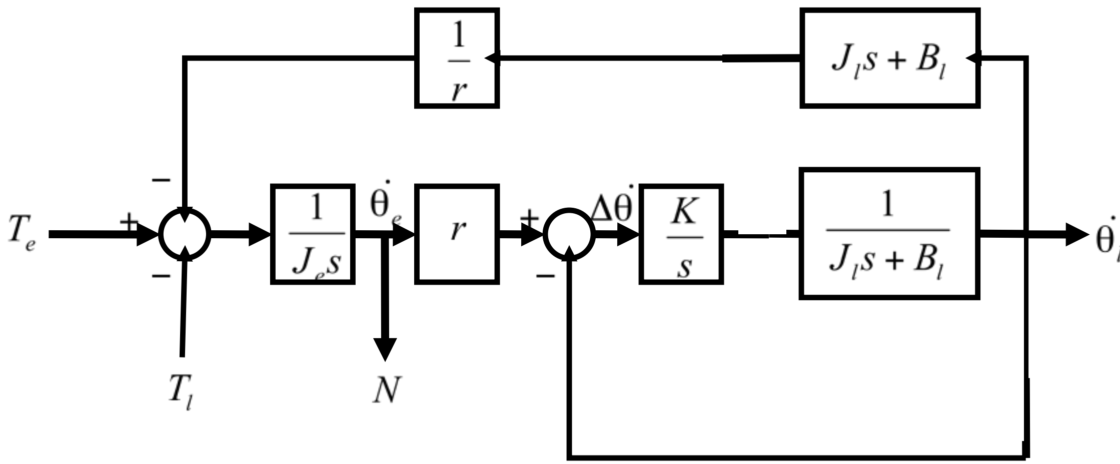

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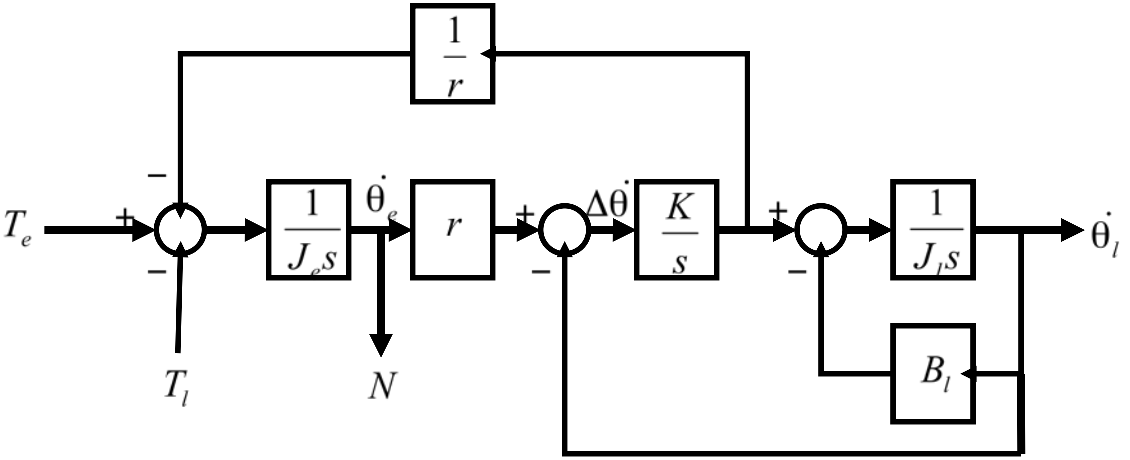

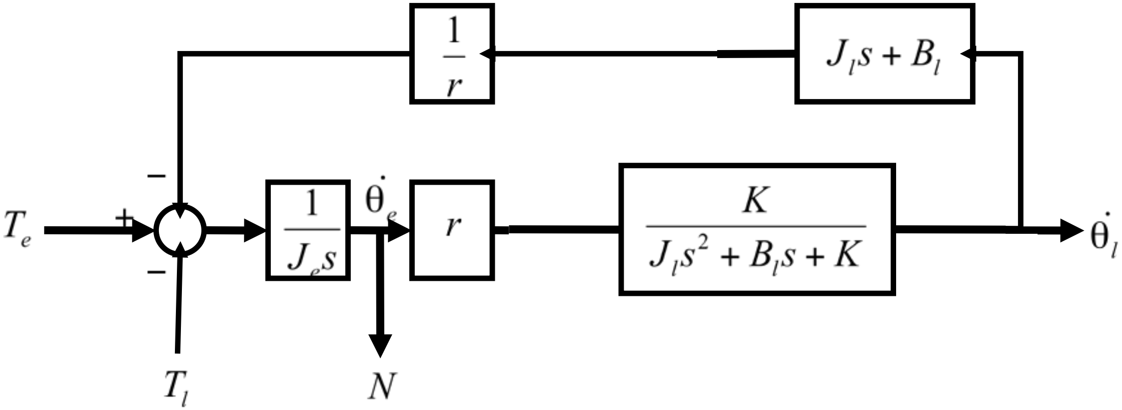

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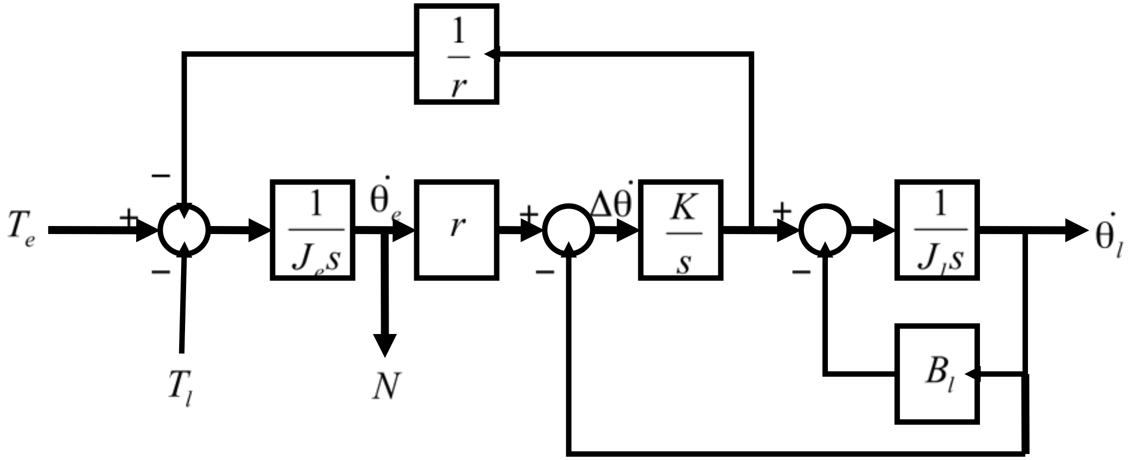

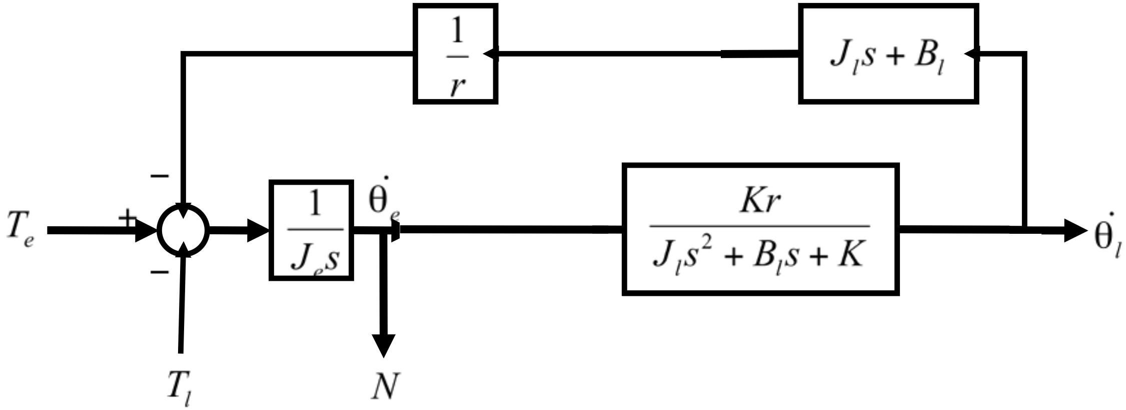

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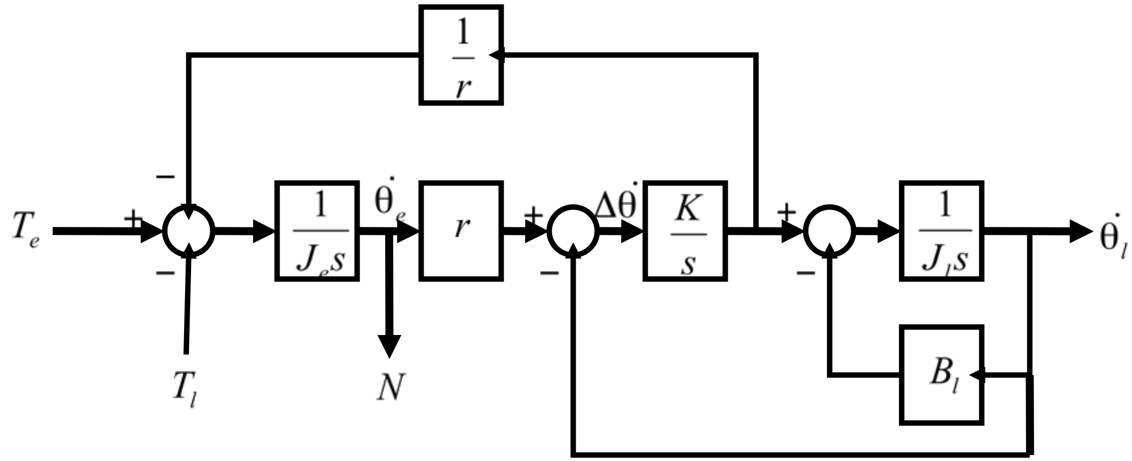

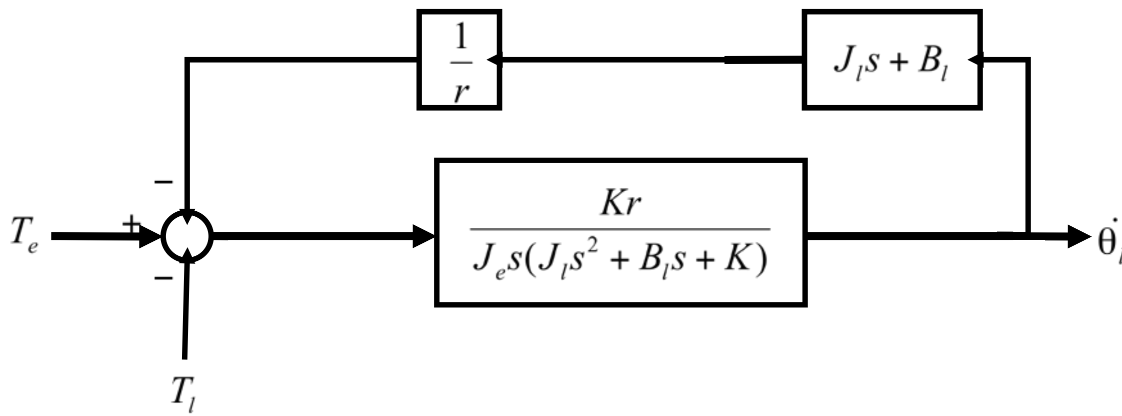

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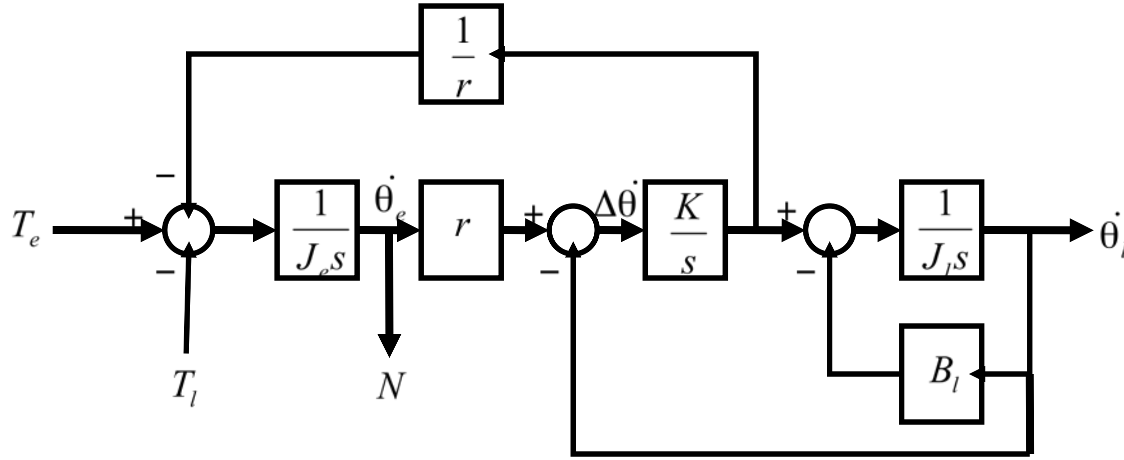
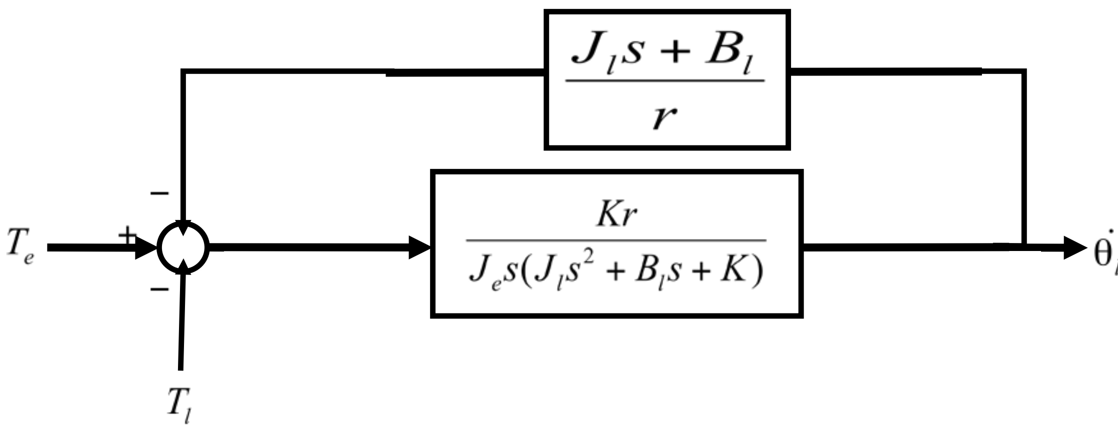




$$H_{\theta_l T_e}(s) = \frac{Kr}{J_e J_l s^3 + J_e B_l s^2 + (J_e K + K J_l)s + K B_l}$$

Sketching the Bode Plot for a Transfer Function (1/2)

Evaluate transfer function on imaginary axis

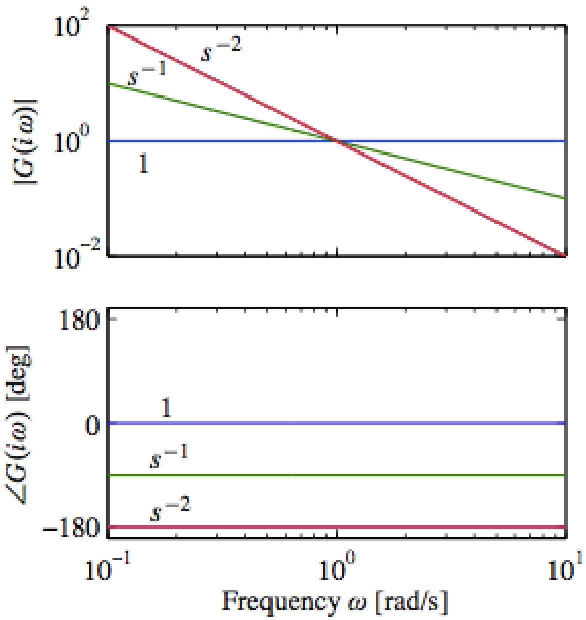
$$M = |G(i\omega)|, \qquad \varphi = \arctan \frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}$$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a \\ \log a - \log\omega & \text{if } \omega > a, \end{cases}$$

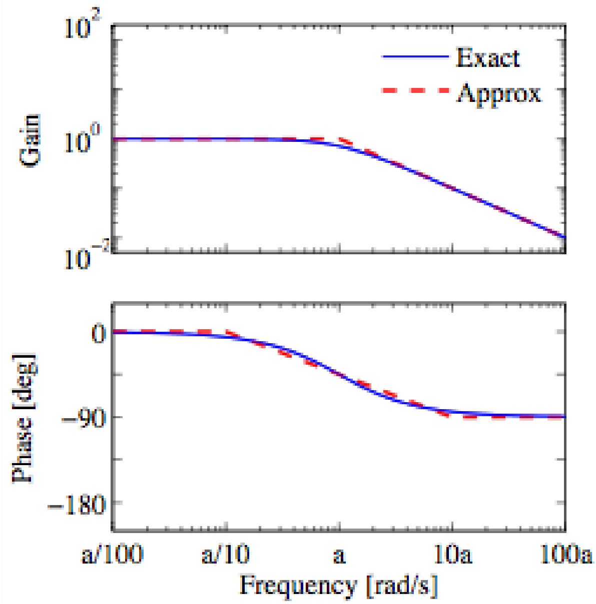
$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10 \\ -45 - 45(\log\omega - \log a) & a/10 < \omega < 10a \\ -90 & \text{if } \omega > 10a. \end{cases}$$

- Plot gain (M) on log/log scale
- Plot phase (φ) on log/linear scale
- Piecewise linear approximations available

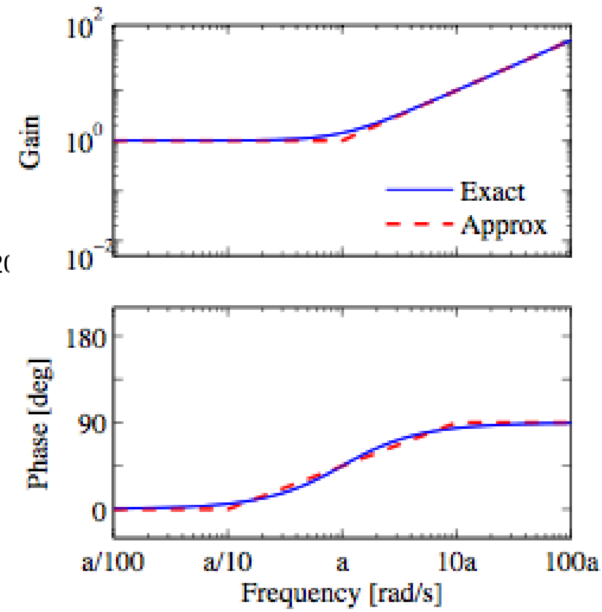
$$G(s) = \frac{1}{s^k}$$



$$G(s) = \frac{a}{s+a}$$



$$G(s) = \frac{s+a}{a}$$



Sketching the Bode Plot for a Transfer Function (2/2)

Complex poles $G(s) = \dfrac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0, \end{cases}$$

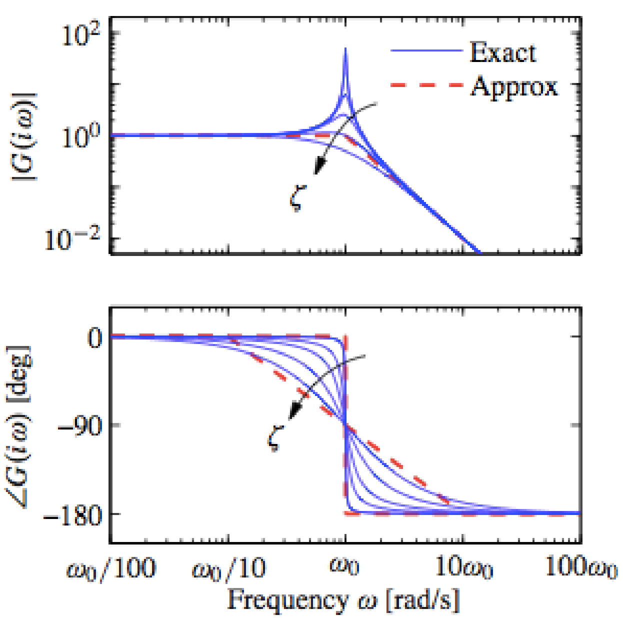
$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}$$

Ratios of products $G(s) = \dfrac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$

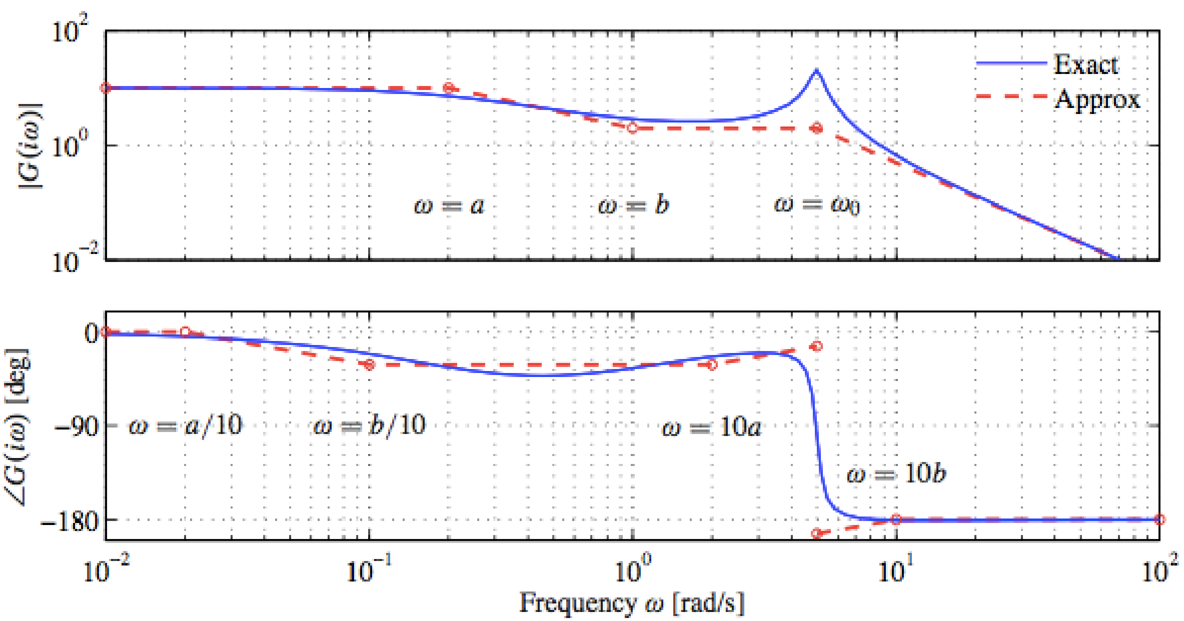
$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|$$

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s)$$

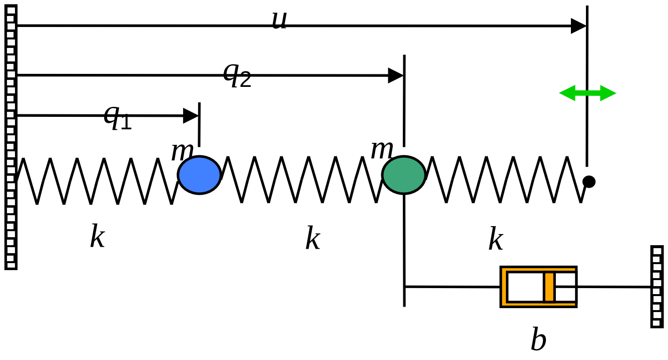
$$G(s) = \frac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$$



$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \qquad a \ll b \ll \omega_0.$$



Example: Coupled Masses



$$H_{q_1 f} = \frac{0.04}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

Poles (Hq1f and Hq2f)

- -0.0200 ± 0.7743j
- -0.0200 ± 0.4468j

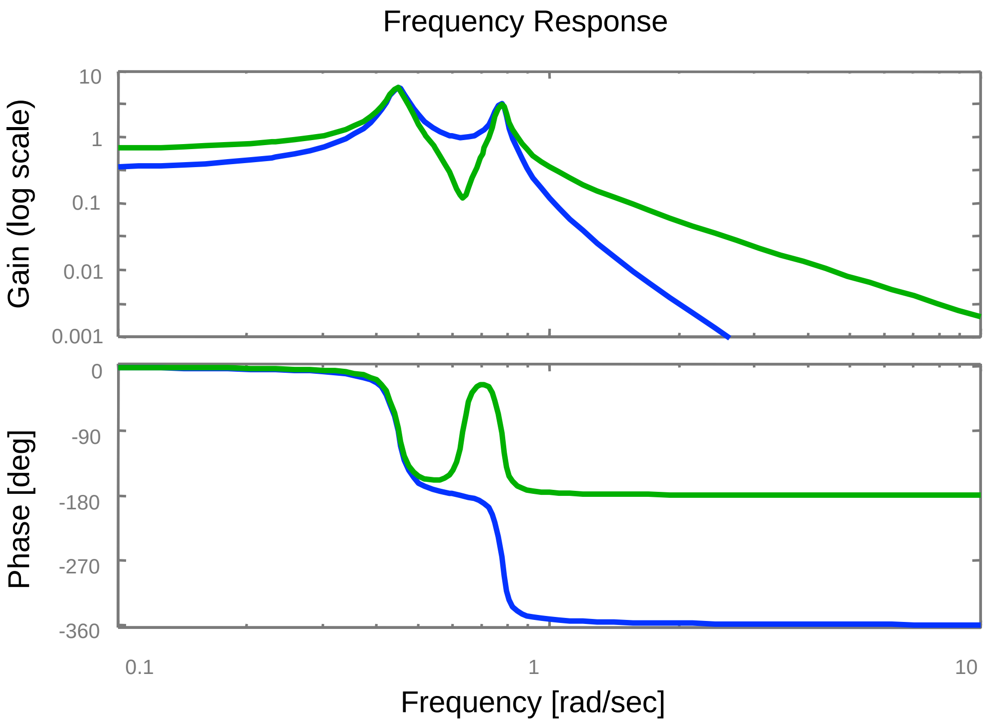
Zeros (Hq2f)

- -0.0200 ± 0.6321j

Interpretation

- Zeros in Hq2f give low response at ω ≈ 0.6321

MATLAB: bode(H)



MATLAB/Python manipulation of transfer functions

Creating transfer functions

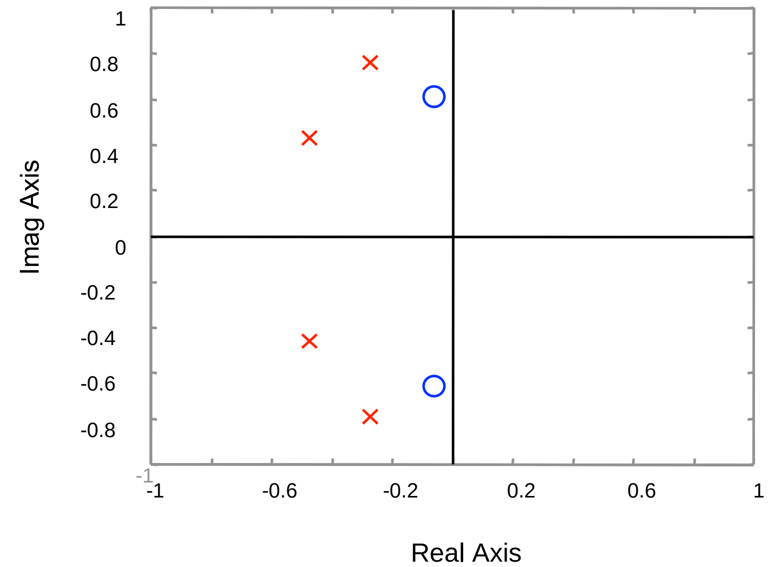
- [num, den] = ss2tf(A, B, C, D)
- sys = tf(num, den)
- num, den = [1 a b] → s2 + as + b

Interconnecting blocks

- sys= series(sys1, sys2), parallel, feedback

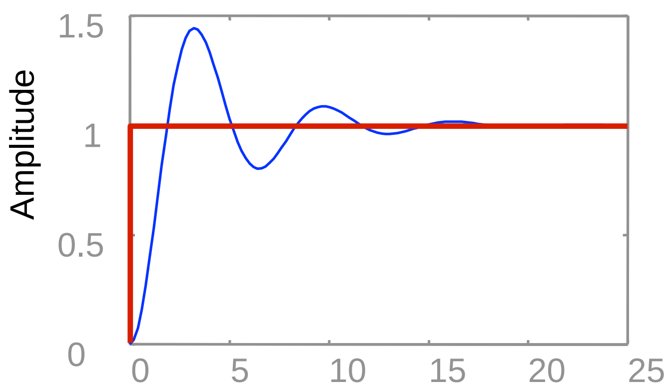
Computing poles and zeros

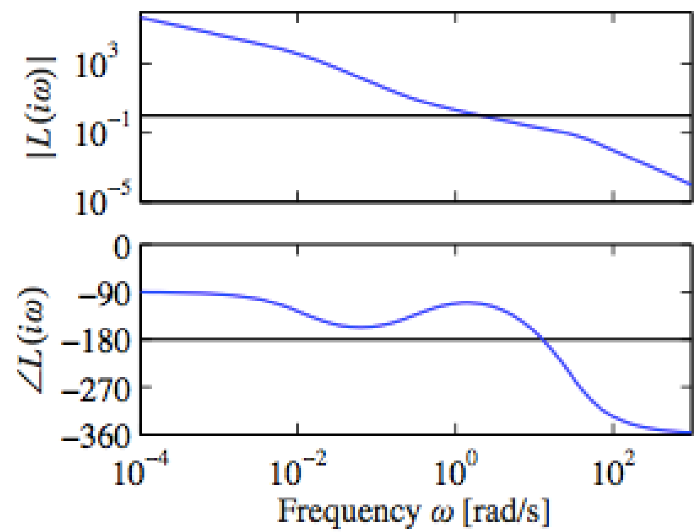
- pole(sys), zero(sys)
- pzmap(sys)



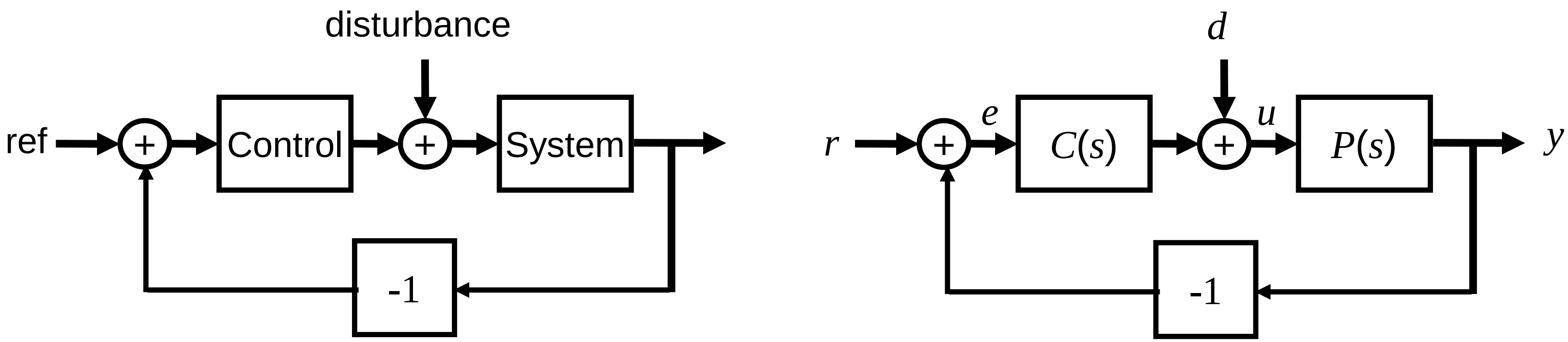
I/O response

- initial(sys)
- step(sys)
- lsim(sys)
- bode(sys)





Control Analysis and Design Using Transfer Functions



Transfer functions provide a method for "block diagram algebra"

- Easy to compute transfer functions between various inputs and outputs
  - Her(s) is the transfer function between the reference and the error
  - Hed(s) is the transfer function between the disturbance and the error

Transfer functions provide a method for performance specification

- Since transfer functions provide frequency response directly, it is convenient to work in the "frequency domain"
  - $H_{er}(s)$ should be small in the frequency range 0 to 10 Hz (good tracking)

Key idea: perform all analysis and design for linear systems in "frequency domain"

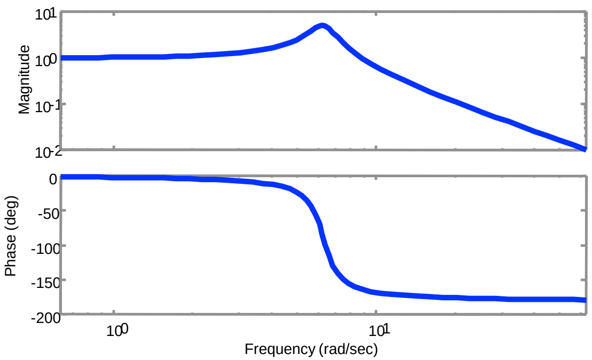
- Convert specifications on time response to specifications on frequency response
- "Shape" the frequency response by design of controller transfer function (Ch 10-13)

Summary: Frequency Response & Transfer Functions

$$u = A\sin(\omega t)$$

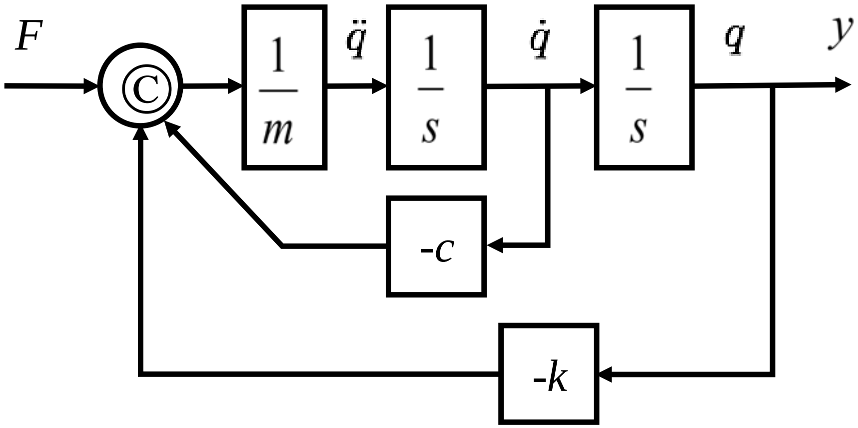
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

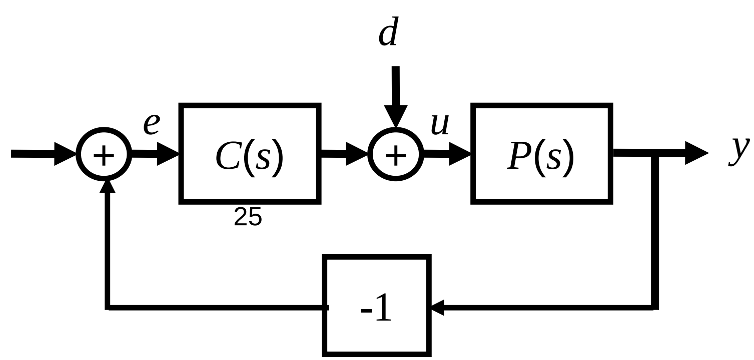
$$y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$$



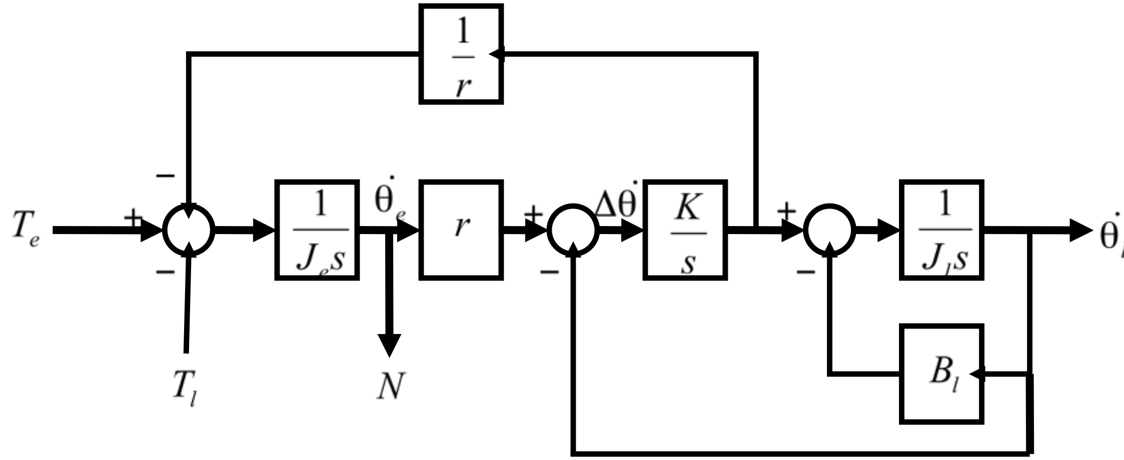
$$G(s) = C(sI - A)^{-1}B + D$$

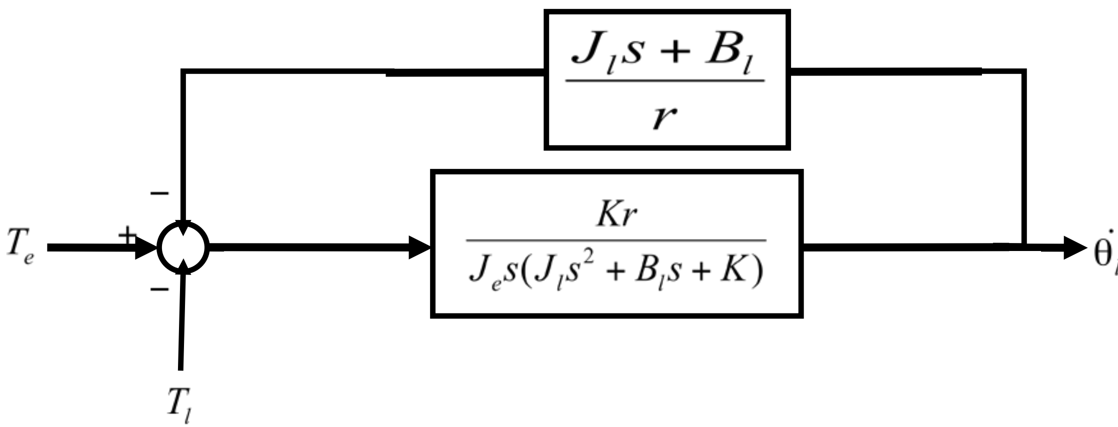
$$G_{y_2 u_1} = G_{y_2 u_2} G_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$





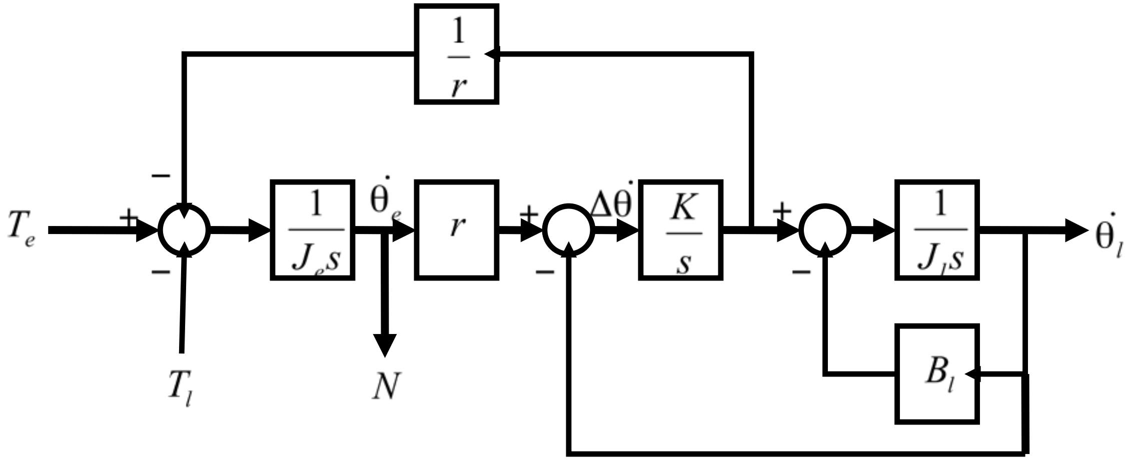
Example: Engine Control of a GM Astro

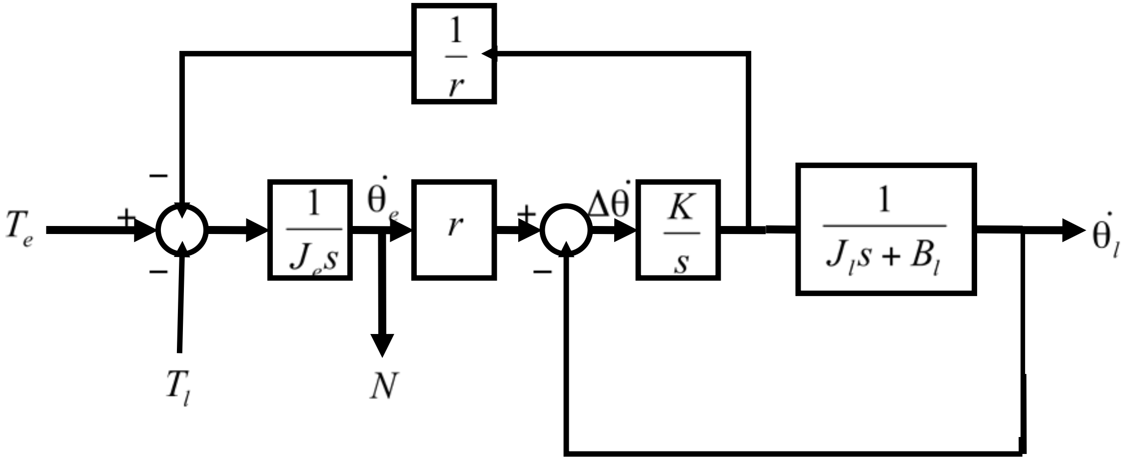




$$H_{\theta_l T_e}(s) = \frac{Kr}{J_e J_l s^3 + J_e B_l s^2 + (J_e K + K J_l)s + K B_l}$$

Example: Engine Control of a GM Astro





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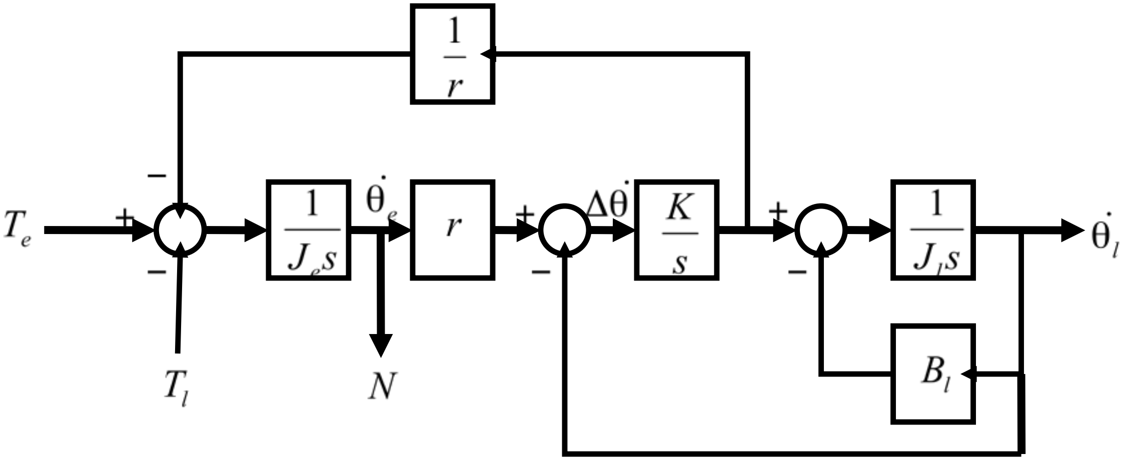

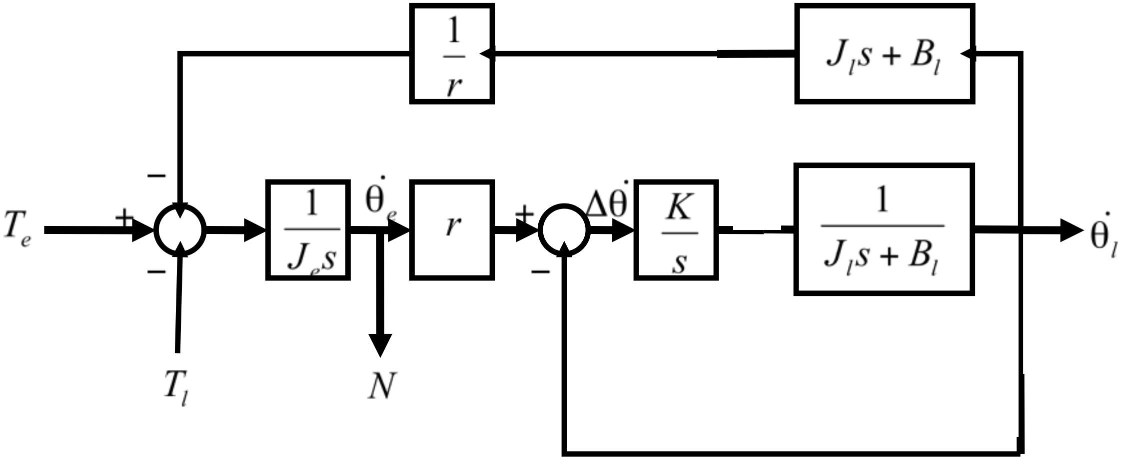

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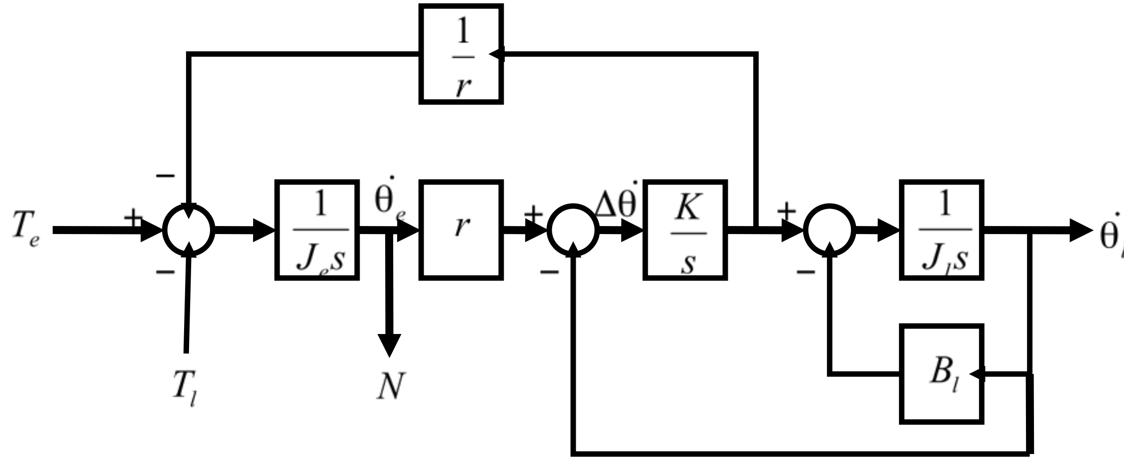

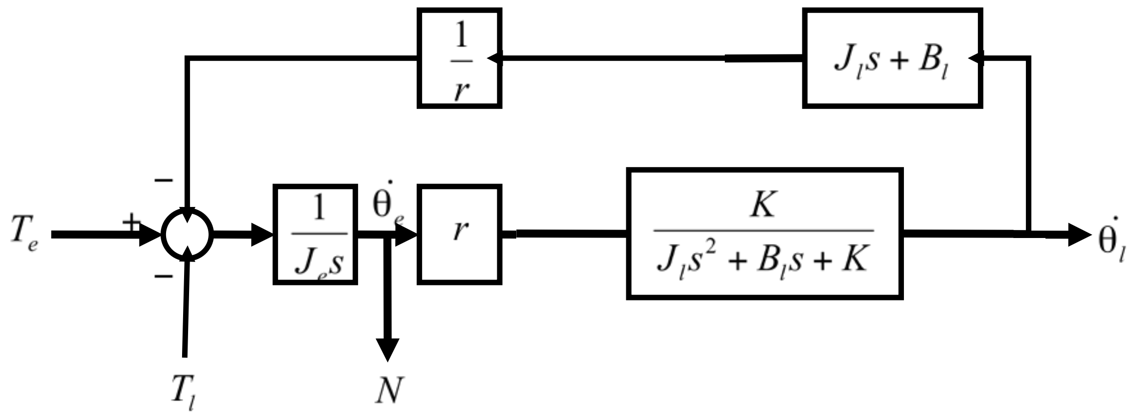

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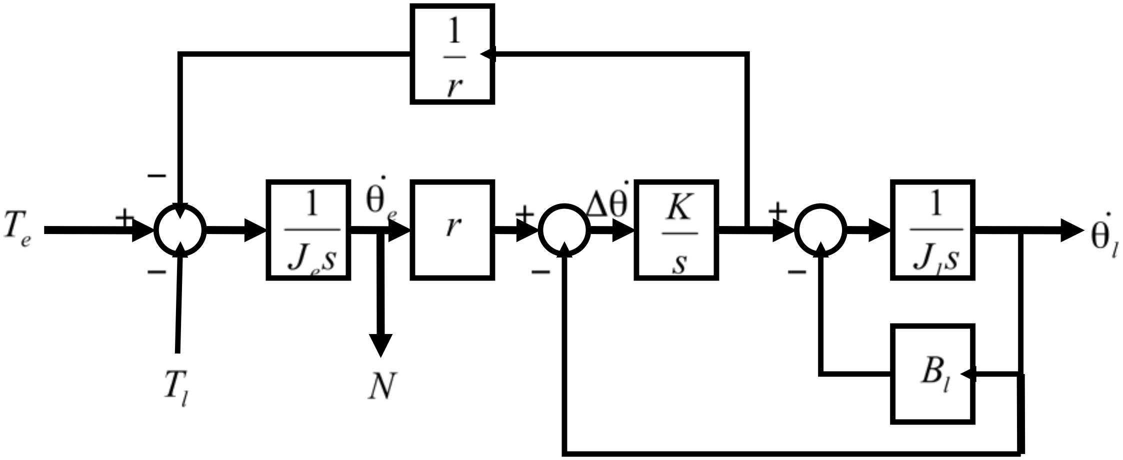

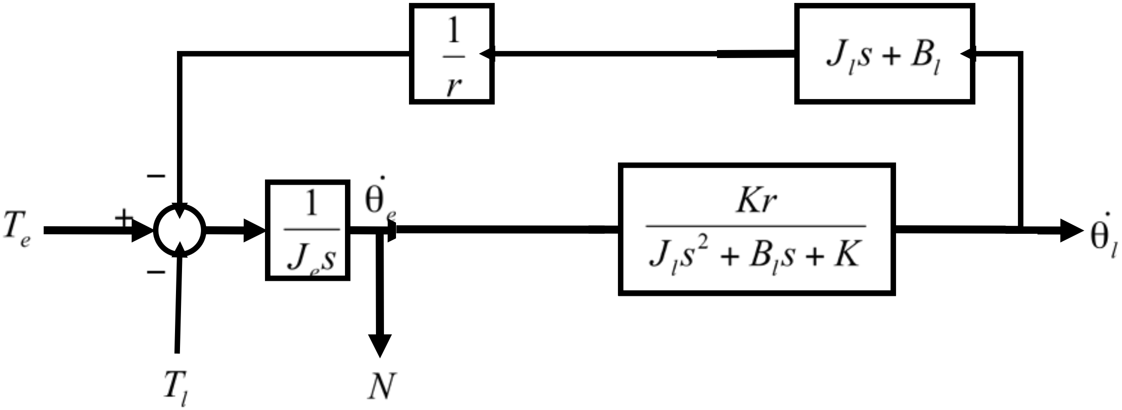

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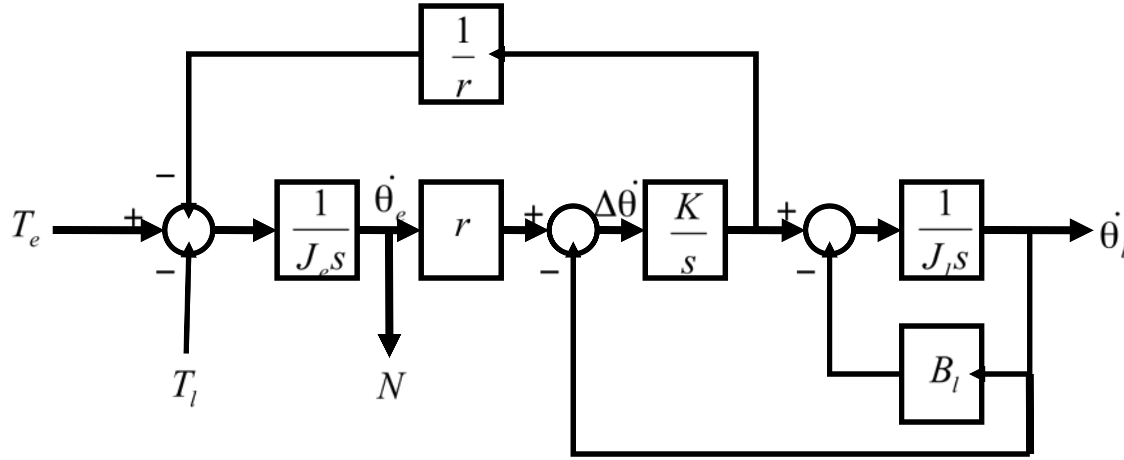

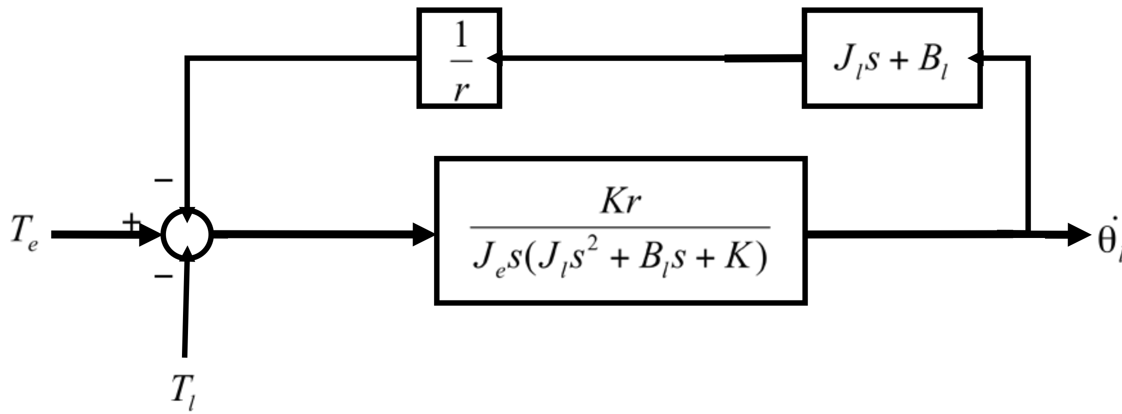

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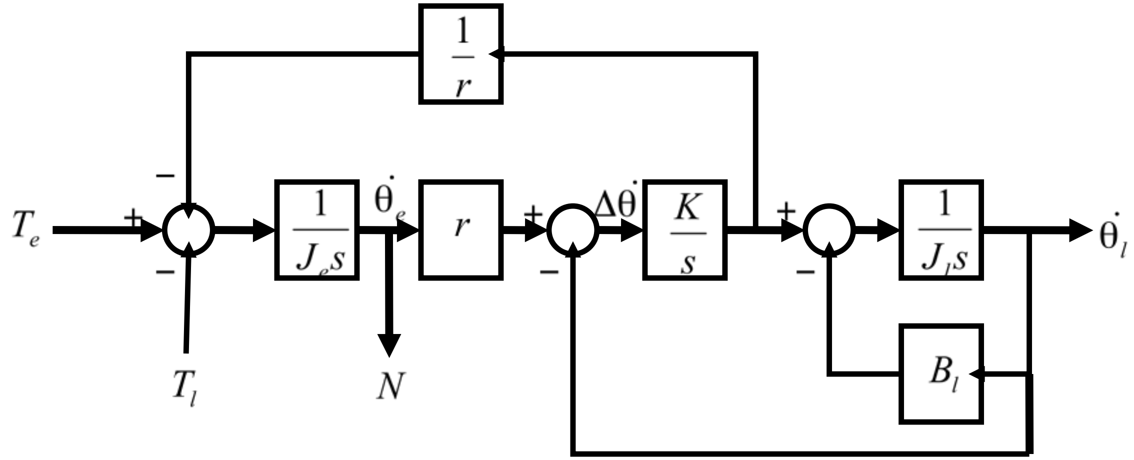

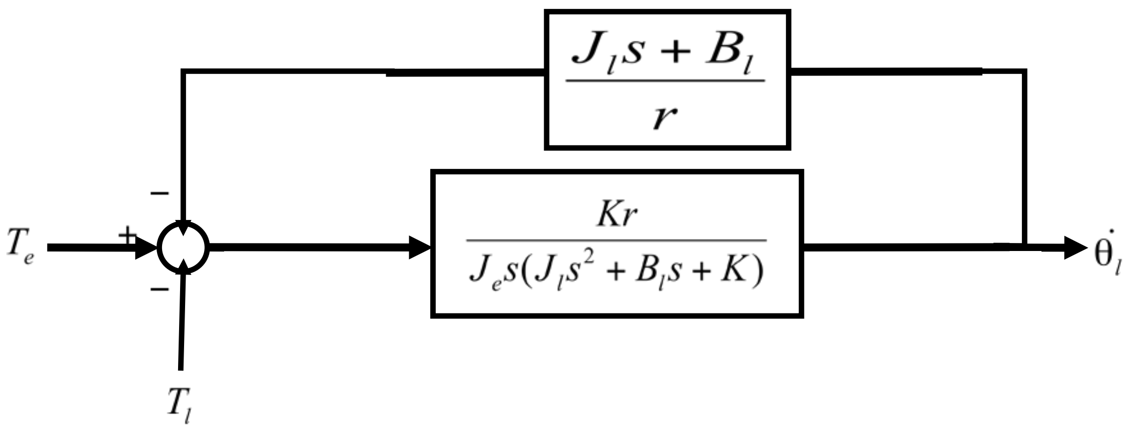

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