

A Symbolic Formula for a C-obstacle

1 Background

Let the robot \mathcal{A} and the obstacle \mathcal{O} be *planar convex polygons*. Our goal in this handout is to derive a closed-form formula for the boundary of the c-obstacle \mathcal{CO} . That is, to find a continuous real-valued function $d(q)$ on c-space, $d : \mathcal{C} \rightarrow \mathbb{R}$, such that

$$\text{bdy}(\mathcal{CO}) = \{q \in \mathcal{C} : d(q) = 0\} \quad \text{where } q = (x, y, \theta).$$

where $\text{bdy}(\mathcal{S})$ denotes the boundary of \mathcal{S} . As discussed below, the boundary of \mathcal{CO} consists of smooth two-dimensional patches that meet along curved or straight edges. Let $n_{\mathcal{A}}$ and $n_{\mathcal{O}}$ be the number of vertices of \mathcal{A} and \mathcal{O} , respectively. From the formula presented below we shall see that $\text{bdy}(\mathcal{CO})$ consists of $2n_{\mathcal{A}}n_{\mathcal{O}}$ patches. The formula will be union of $2n_{\mathcal{A}}n_{\mathcal{O}}$ terms, each representing one patch. Each patch term will be in turn be defined by the intersection of several inequalities.

Recall that $\mathcal{A}(q)$ is the set of points occupied by the robot when placed at configuration q . Each two-dimensional patch comprising the boundary of \mathcal{CO} corresponds to one of two generic types of contact.

- **Type EV contact:** Configurations q where an edge of the robot, denoted $E_i^{\mathcal{A}}(q)$, contacts a fixed vertex o_j of the obstacle \mathcal{O} . These contacts are often called *Type A* contacts in the robotics motion planning literature, in deference to nomenclature use in the original paper on configuration space by Lozano-Perez, Mason, and Taylor.
- **Type VE contact:** Configurations q where a vertex $a_i(q)$ of \mathcal{A} contacts a fixed edge $E_j^{\mathcal{O}}$ of \mathcal{O} . These contacts are often called *Type B* contacts in the literature.

The fixed- θ slice of each patch is a straight-line segment when both robot and obstacle are polygons. As the robot rotates, the segments of a Type EV patch rotate (and therefore a Type EV patch is a “ruled surface”), and the segments of a Type VE patch translate parallel to the associated edge of \mathcal{O} .

The curves separating neighboring patches are of two types. Those that lie in a fixed- θ slice correspond to edge-edge contact. Those that lie transversally to the fixed- θ slices correspond to vertex-vertex contact. The vertex-vertex ones have the property that their projection onto the (x, y) -plane is a circular arc. See Figure 1.

2 The Formula

First we derive a formula for a Type EV patch. Consider the set of all configurations q where the edge $E_i^{\mathcal{A}}(q)$ touches the vertex o_j , such that the interiors of \mathcal{A} and \mathcal{O} do not

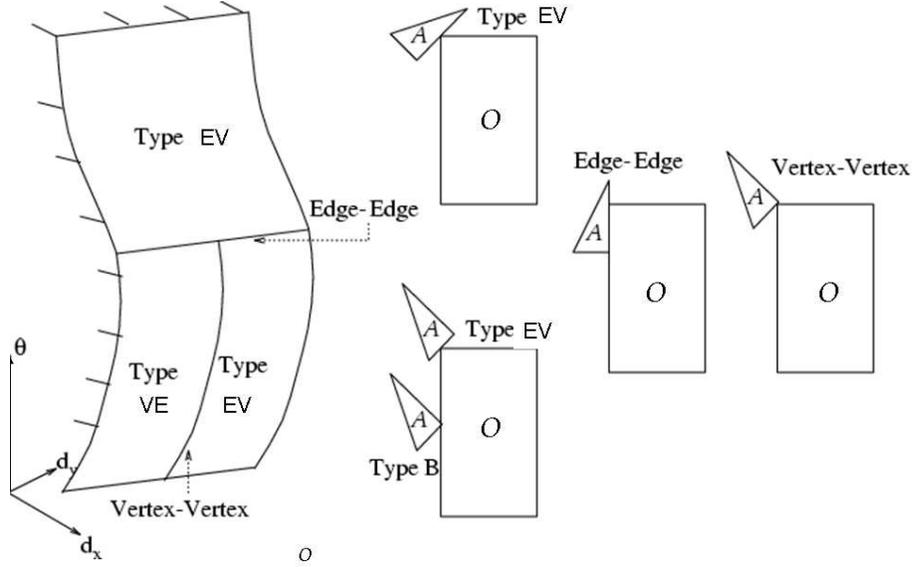


Figure 1: Type EV and Type VE patches

overlap (Figure 2(a)). Let l be the line underlying the edge $E_i^A(q)$. The equation of l is $\nu_i^A(q) \cdot (x - a_i(q)) = 0$. The first constraint is that o_j must lie on l :

$$\nu_i^A(q) \cdot (o_j - a_i(q)) = 0. \quad (1)$$

This defines the smooth two-dimensional surface underlying the patch. Second, regardless of where o_j lies on l , the range of allowed orientations for \mathcal{A} such that its interior does not overlap the interior of \mathcal{O} is determined as follows. The allowed orientations are exactly those where the vertices o_{j-1} and o_{j+1} lie in the halfspace bounded by l which does not contain \mathcal{A} . In formulas:

$$\nu_i^A(q) \cdot (o_{j-1} - a_i(q)) \geq 0$$

and

$$\nu_i^A(q) \cdot (o_{j+1} - a_i(q)) \geq 0.$$

The intersection of these inequalities with (1) gives the “upper” and “lower” edges of the patch, corresponding to edge-edge contacts. Last, o_j must lie within the edge $E_i^A(q)$. Convince yourself that this is captured by the following two inequalities:

$$(o_j - a_i(q)) \cdot (a_{i+1}(q) - a_i(q)) \geq 0$$

and

$$(o_j - a_i(q)) \cdot (a_{i+1}(q) - a_i(q)) \leq \|E_i^A\|^2.$$

($\|E_i^A\|$ is the length of the edge $E_i^A(q)$. It does not depend on q). These inequalities determine the “left” and “right” edges of the patch, corresponding to vertex-vertex contacts. The patch

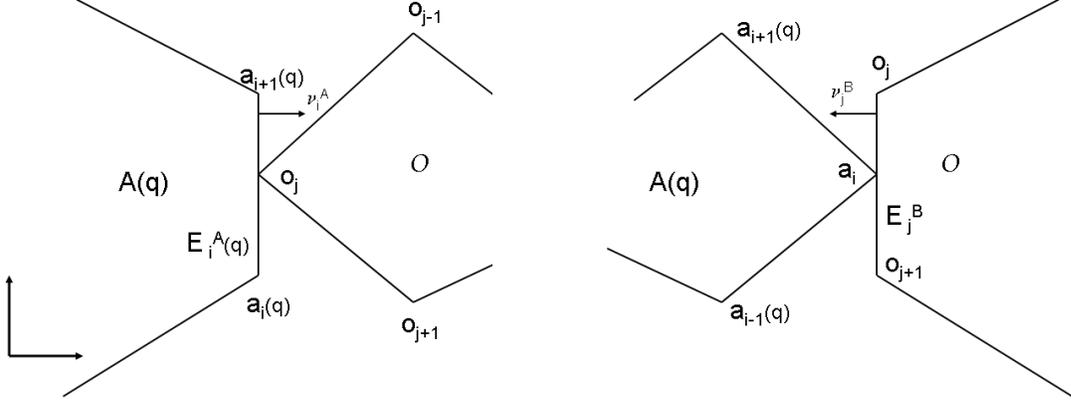


Figure 2: Type EV contact (left), and Type VE contact (right)

is described by the conjunction (equivalently, intersection of the respective sets) of these five terms.

Consider now a Type VE patch, where the vertex $a_i(q)$ touches the edge $E_j^{\mathcal{O}}$, such that the interiors of \mathcal{A} and \mathcal{O} do not overlap (Figure 2(b)). Convince yourself that the conjunction of the following five terms exactly characterizes the patch:

$$\nu_j^{\mathcal{O}} \cdot (a_i(q) - o_j) = 0,$$

which is the surface underlying the patch,

$$\nu_j^{\mathcal{O}} \cdot (a_{i-1}(q) - o_j) \geq 0 \quad \text{and} \quad \nu_j^{\mathcal{O}} \cdot (a_{i+1}(q) - o_j) \geq 0,$$

which are the upper and lower edges,

$$(a_i(q) - o_j) \cdot (o_{j+1} - o_j) \geq 0, \quad \text{and} \quad (a_i(q) - o_j) \cdot (o_{j+1} - o_j) \leq \|E_j^{\mathcal{O}}\|^2,$$

which are the left and right edges.

In total we have

$$\text{bdy}(\mathcal{C}\mathcal{O}) = \left(\bigcup_{1 \leq i \leq n_{\mathcal{A}}, 1 \leq j \leq n_{\mathcal{O}}} (\text{Type EV patch}) \right) \cup \left(\bigcup_{1 \leq i \leq n_{\mathcal{A}}, 1 \leq j \leq n_{\mathcal{O}}} (\text{Type VE patch}) \right). \quad (2)$$

where the shorthand notation ‘‘Type EV patch’’ stands for the set of equalities and inequalities that define a Type EV patch. There are exactly $2n_{\mathcal{A}}n_{\mathcal{O}}$ terms, which is the number of distinct patches comprising the boundary of $\mathcal{C}\mathcal{O}$.

The continuous function, $d(q)$, that implicitly describes the boundary can be defined as follows. Given two sets described by $\mathcal{S}_1 = \{q : \alpha(q) \leq 0\}$ and $\mathcal{S}_2(q) = \{q : \beta(q) \leq 0\}$, $\mathcal{S}_1 \cap \mathcal{S}_2$

can be replaced by the inequality $\max\{\alpha(q), \beta(q)\} \leq 0$ and $\mathcal{S}_1 \cup \mathcal{S}_2$ can be replaced by $\min\{\alpha(q), \beta(q)\} \leq 0$. Since a patch is defined as the intersection of inequalities and the c-obstacle boundary is defined as the union of patches, Equation (2) can be constructed from nested sequences of min and max operations.