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Linearization in the large of nonlinear systems and Koopman operator spectrum

Yueheng Lan [a], Igor Mezić [b,c,*]

[a] Department of Physics, Tsinghua University, Beijing 100084, PR China

[b] The Center for Control, Dynamical Systems and Computation, University of California, Santa Barbara, CA 93106, USA

[c] Department of Mechanical Engineering, University of California, Santa Barbara, 93106, USA

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ABSTRACT

According to the Hartman–Grobman Theorem, a nonlinear system can be linearized in a neighborhood of a hyperbolic stationary point. Here, we extend this linearization around stable (unstable) equilibria or periodic orbits to the whole basin of attraction, for both discrete diffeomorphisms and flows. We discuss the connection of the linearizing transformation to the spectrum of Koopman operator.

1. Introduction

Much progress has been made in the study of nonlinear dynamical systems yet challenges remain, due to the existence of extremely rich orbit structures in the phase space, culminating with the appearance of strange sets with chaotic dynamics. Unveiling the orbit topology and global behavior of a nonlinear system constitutes one major goal of the dynamical systems theory. Although locally (e.g. around equilibrium points) orbits of a nonlinear system can be mapped to those of a linear one [1], it is difficult to state the precise region of validity of this mapping because of the usually limited information on the global phase space structure. Moreover, it is often true that this (local) mapping does not provide a linearizing transformation in the whole phase space, most commonly due to the existence of disconnected attractors and their basins of attraction. The best expectation, therefore, is to see the preservation of the linear system orbit structure in the basin of attraction. Accordingly, the whole phase space can be effectively viewed as a juxtaposition of such domains. Ergodic partition theory [2–5] provides means for detecting such domains. This theory utilizes spectral properties of the so-called Koopman operator [6,7] at eigenvalue 1. Here we show that linearization in the whole basin of attraction is indeed possible for a large class of dynamical systems, and that this fact is also related to spectral properties of the Koopman (or composition) operator.

In the neighborhood of a normally hyperbolic manifold, linearization is always possible [8–10]. Specifically, around a hyperbolic equilibrium, Hartman–Grobman theorem establishes topological conjugacy of the nonlinear system with a linear one [11,12]. With further restrictions, the linearization map can be made C^k-continuous or even smoother [13]. For analytic vector fields satisfying a non-resonance condition, Poincaré–Siegel theorem indicates the existence of an analytic linearization [14]. However, all these theorems only provide a much under-estimated linearization region. Results for linearization in the whole phase space were proved for vector fields with a bounded nonlinear part which are small Lipschitzian [15–17], *i.e.*, for vector fields with weak nonlinearity. Symmetry considerations with the application of Lie algebra provide an alternative way of detecting linearizability of nonlinear ordinary differential equations (ODEs), but it requires detecting all the symmetries of nonlinear ODEs [18,19], which is often a formidable job itself. On the other hand, transformations of various types are designed to linearize nonlinear ODEs with solutions worked out explicitly [20,21]. However, each of these only applies to a restricted class of ODEs.

In most linearization schemes (with the exception of work in [22], where the authors use differential topology techniques to obtain results on global asymptotic exponential stability, without requesting that the spectrum be preserved, the condition that we

* Corresponding author at: Department of Mechanical Engineering, University of California, Santa Barbara, 93106, USA.
E-mail address: mezic@engineering.ucsb.edu (I. Mezić).

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keep in our work), the global results are obtained either through the exact solution of the equation or with the assumption of weak nonlinearity [23–25], so the behavior is dominated by the linear part. For general nonlinear systems, these requirements are usually not satisfied and new approaches are needed. In this paper, we utilize the orbit conjugacy generated by the flow itself to extend the local results of Hartman–Grobman to the whole basin of attraction of a stable equilibrium or a limit cycle, which avoids proof of convergence in a large region. Therefore, our results go far beyond the immediate neighborhood of an equilibrium while the linearization is expectedly restricted to part of the phase space if multiple equilibria exist. We also put linearization schemes in the context of spectral properties of the Koopman operator.

The paper is organized as follows. In the next section, after stating Hartman's local theorem, we prove linearization in the basin of attraction of an equilibrium for maps, autonomous or periodic flows. The linearization in the basin of attraction of a stable periodic orbit is also proved. These theorems can all be applied on the stable or unstable manifold of equilibria or limit cycles. In Section 3, several typical ODEs are used as examples to demonstrate the linearization in large regions, with the help of analytical or numerical integration. In the final section, we summarize our results and point out possible future directions for further investigation.

2. Extension of linearization theorems

2.1. Definitions, notations and Hartman–Grobman theorem

In the following, we will consider a continuously differentiable dynamical system defined in some open region D of $\mathbb{R}^n$,

$$\dot{x} = f(x) = Ax + v(x), \tag{1}$$

where the origin $x = 0$ is a stationary point contained in D, $A = Df(0)$ is the gradient of the vector field at $x = 0$, and $v(x) \sim O(|x|^2)$ is the nonlinear part of the vector field. System (1) induces a flow $\phi(x, t) : D \times \mathbb{R} \to D$ and the positively invariant basin of attraction Ω of the fixed point is defined as usual

$$\Omega = \{x : \phi(x, t) \in D,\ \forall t \geq 0 \text{ and } \lim_{t\to\infty} \phi(x, t) = 0\}. \tag{2}$$

B_r is used to denote a ball of radius r in $\mathbb{R}^n$. For two vectors $a, b \in \mathbb{R}^n$ the inner product is defined as

$$a \cdot b = \sum_{i=1}^{n} a_i b_i,$$

where a_i, b_i are components of vectors a and b. The 2-norm of a vector b is defined as

$$\|b\|_2 = \sqrt{b \cdot b}.$$

We use capital letters to denote matrices, like matrix A in Eq. (1) and I for the identity matrix. All eigenvalues of a symmetric real matrix Q are real. We use $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ to indicate its largest and smallest eigenvalue, respectively. It is also convenient to define the 2-norm of a matrix P

$$\|P\|_2 = \max_{\|x\|_2=1} \|Px\|_2 .$$

If P is symmetric and positive definite, $\|P\|_2 = \lambda_{\max}(P)$.

Here, we present the most well-known local theorem on the linearization of a nonlinear vector field—the Hartman–Grobman Theorem, which states that provided the nonlinear system (1) is hyperbolic at the origin, then it is locally conjugate to the linear system

$$\dot{x} = Ax. \tag{3}$$

For convenience, we reproduce the theorem as follows

Theorem 2.1 (*Hartman–Grobman Theorem*). *Let* $\mathbf{f} \in C^1(D)$. *Suppose that A has no eigenvalue with zero real part. Then there exists a homeomorphism h of an open set $U \subset D$, $0 \in U$ onto an open set $V \subset \mathbb{R}^n$, $0 \in V$ such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$,*

$$h \circ \phi(x_0, t) = e^{At} h(x_0); \tag{4}$$

i.e., h maps trajectories of (1) near the origin to trajectories of the linear system (3) and preserves the parametrization by time.

Note that the Koopman operator U^t corresponding to (1), acting on functions $g : D \to \mathbb{C}$ is defined as

$$U^t g(x) = g(\phi(x, t)).$$

A function φ is called an eigenfunction of U^t associated with eigenvalue λ, provided

$$U^t \varphi(x) = \exp(\lambda t)\varphi(x). \tag{5}$$

In the same vein, we could call a matrix A an eigenmatrix of U^t associated with eigenmapping h provided (4) holds. Note that within Hartman–Grobman theorem and the Hartman Theorem stated below, this is the case only *locally*, around an equilibrium point, and possibly for finite time. Let A have distinct real eigenvalues, and it can be transformed into a diagonal matrix Λ using a linear transformation V. In that case setting $z = V^{-1}x$ leads to

$$\dot{z} = V^{-1}AVz = \Lambda z,$$

and V is the matrix of column (right) eigenvectors, then we get

$$V^{-1}h \circ \phi(x_0, t) = V^{-1}e^{At}h(x_0),$$

and $k = V^{-1}h$ satisfies

$$k \circ \phi(x_0, t) = e^{\Lambda t}k(x_0); \tag{6}$$

i.e. each component function of k is an eigenfunction of U^t. In the theorem, $h(x)$ is a homeomorphism. Additional conditions may set h to be diffeomorphic or even analytic. For a stable or an unstable equilibrium, we have the following Hartman Theorem.

Theorem 2.2 (*Hartman*). *Let $f \in C^2(D)$. If all of the eigenvalues of the matrix A have negative (or positive) real part, then there exists a C^1-diffeomorphism $h = x + \tilde{h}$ of a neighborhood $U \in D$ of $x = 0$ onto an open set V containing the origin such that for each $x \in U$ there is an open interval $I(x) \subset \mathbb{R}$ containing zero such that for all $x \in U$ and $t \in I(x)$*

$$h \circ \phi(x, t) = e^{At}h(x).$$

In addition,

$$\lim_{x\to 0} \frac{\left\|\tilde{h}(x)\right\|_2}{\|x\|_2} = 0.$$

In Hartman's version of the theorem, the time interval can be extended to $+\infty$ for stable dynamics and $-\infty$ for unstable dynamics.

In the following, we will extend Hartman's local theorem for stable equilibria to a global one that is valid in the whole basin of attraction.

2.2. Linearization of autonomous flows

Before proving the main theorem, we prove two lemmas.

Lemma 2.1 (*Differentiable Dependence on the Arguments*). *Consider the flow*

$$x = \phi_1(x_0, t), \qquad h = \phi_2(x_0, h_0, t)$$

defined by the ODEs

$$\dot{x} = f(x), \qquad \dot{h} = g(x, h),$$

where $x, h \in \mathbb{R}^n, f \in C^1(\mathbb{R}^n), g \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $x_0 = \phi_1(x_0, 0), h_0 = \phi_2(x_0, h_0, 0) = h_0(x_0) \in C^1(\Sigma)$, where $\Sigma \subset \mathbb{R}^n$ is a simple smooth $(n-1)$-dimensional hyper-surface. Let $D \subset \mathbb{R}^n$ be an open region. If for $x_1 \in D$, there exists $x_0(x_1) \in \Sigma, t(x_1) \in \mathbb{R}$ such that $x_1 = \phi_1(x_0(x_1), t(x_1))$ and the orbit passing x_1 intersects Σ transversely, then the following is true,

(i) *$\exists B_r \subset D$ with $x_1 \in B_r$, such that $\forall x \in B_r, \exists x_0(x) \in \Sigma, t(x) \in \mathbb{R}$ such that $x = \phi_1(x_0(x), t(x))$ and the orbit passing x intersects Σ transversely;*
(ii) *$x_0(x), t(x) \in C^1(B_r)$;*
(iii) *$h(x) = \phi_2(x_0(x), h_0(x_0(x)), t(x)) \in C^1(B_r)$.*

Proof. According to the basic theorems of ODEs, $f, g \in C^1(\mathbb{R}^n)$ implies that ϕ_1, ϕ_2 are C^1 functions of their parameters.

(i) and (ii) follow the C^1-continuous dependence of $\phi_1(x, t)$ on its arguments and the implicit function theorem.

(iii) originates from the continuity of combination of maps, since $\phi_2(x_0, h_0, t) \in C^1, h_0(x_0) \in C^1(\Sigma)$ and $x_0(x), t(x) \in C^1(B_r)$ as proved in assertion (ii). □

The next lemma uniquely labels each point in the basin of attraction (repulsion) by its initial point on a closed surface and the evolution time along an orbit.

Lemma 2.2 (*Labeling Lemma*)**.** *Consider the dynamical system Eq. (1) with $f(x) \in C^1(D)$. $\Sigma \subset D$ is a simple closed C^1 manifold of dimension $n-1$ enclosing the origin and $f(x) \cdot n_\perp(x) \neq 0, \forall x \in \Sigma$, where $n_\perp(x)$ is the outward normal to Σ at x. Let $\phi(x_0, t)$ be the flow induced by Eq. (1) with $x_0 = \phi(x_0, 0)$. Let $\Omega = \cup_{x\in\Sigma} l_x$ where $l_x = \phi(x, (t_-(x), t_+(x))) \subset D$ is the integral curve that passes $x \in \Sigma$ with $t_- < t_+$. Then, $\forall x \in \Omega$, there exist a unique $x_0(x) \in \Sigma$ and $t(x) \in (t_-(x_0), t_+(x_0))$, such that $x = \phi(x_0(x), t(x))$.*

Proof. (i) The closed surface Σ cuts Ω to two disjoint parts Ω_- and Ω_+, *i.e.*, $\Omega = \Omega_- \cup \Sigma \cup \Omega_+$. For any continuous curve $l \subset \Omega$ that connects $x_- \in \Omega_-$ and $x_+ \in \Omega_+$, there exists one point $x_0 \in l$ that lies on Σ since Σ is closed.

(ii) Any integral curve in Ω intersects Σ exactly once.

According to the definition of Ω, all the points in Ω lie on the integral curves that cross Σ. So, every integral curve intersects Σ at least once. Suppose $\exists x_1 \in \Sigma$, so that l_{x_1} intersects Σ more than once. Without loss of generality, we choose $t_1 \neq 0$ with $|t_1|$ being the smallest to represent the immediate neighboring intersection in time. As $f \cdot n_\perp \neq 0$, l_{x_1} must cross Σ with an angle other than zero, and thus $f(x_1) \cdot n_\perp(x_1)$ and $f(\phi(x_1, t_1)) \cdot n_\perp(\phi(x_1, t_1))$ must have opposite signs, indicating the inward and outward movement of the trajectory. However, $f(x) \cdot n_\perp(x)$ is a continuous function of x on any continuous curve $\gamma \subset \Sigma$ that connects x_1 and $\phi(x_1, t_1)$. Therefore, $\exists x_2 \in \gamma$ such that $f \cdot n_\perp|_{x=x_2} = 0$, in contradiction to our assumption. So, the assertion at the beginning is proved.

(iii) $l_x \cap l_{x'} = \emptyset$ whenever $x \neq x'$.

If this is not the case, then $l_x \cup l_{x'} \subset \Omega$ will merge into one integral curve intersecting Σ at least twice, which contradicts (ii).

(iv) $\forall x_1 \in \Sigma, t_1, t_2 \in (t_-(x_1), t_+(x_1))$, if $t_1 \neq t_2$, then $\phi(x_1, t_1) \neq \phi(x_1, t_2)$.

Otherwise, l_{x_1} would be a periodic orbit and transversely intersect Σ at least twice, which is a contradiction to (ii).

(v) So, $\forall x \in \Omega$ and $x \neq 0$, from the definition of Ω, $x \in l_{x_0}$ for some $x_0 \in \Sigma$. According to (iii), x_0 is unique. Also, $\exists t(x) \in (t_-(x), t_+(x))$, such that $x = \phi(x_0(x), t(x))$. According to (iv), $t(x)$ is unique. We proved the lemma. □

Remark. The above proof remains valid even if $t_-(x) = -\infty$ or $t_+(x) = +\infty$, which is usually the case if the whole basin of attraction is enclosed in the domain D. On the other hand, if D includes only part of the basin or the integration time is limited by appearance of singularity or other factors, $t_-(x)$ or $t_+(x)$ could be finite.

We are now ready to prove a linearization theorem for a flow in the basin of attraction of a stable equilibrium.

Theorem 2.3 (*Autonomous Flow Linearization*)**.** *Consider the system (1) with $v(x) \in C^2(D)$. Assume that A is a $n \times n$ Hurwitz matrix,* i.e., *all its eigenvalues have negative real parts. So, $x = 0$ is exponentially stable and let Ω be its basin of attraction. Then $\exists h(x) \in C^1(\Omega) : \Omega \to \mathbb{R}^n$, such that $y = a(x) = x + h(x)$ is a C^1 diffeomorphism with $\mathbf{D}a(0) = I$ in Ω and satisfies $\dot{y} = Ay$.*

Proof. (i) A is a Hurwitz matrix, so for any positive definite symmetric matrix Q, the Lyapunov equation

$$PA + A^T P = -Q$$

has a unique solution P that is positive definite and symmetric as well [26].

(ii) According to Hartman's theorem, there exists a C^1 diffeomorphism $y(x) = x + \tilde{h}(x)$ in a neighborhood $\mathcal{N}_1$ of $x = 0$, such that

$$\dot{y} = Ay, \quad \text{and} \quad \frac{\left\|\tilde{h}(x)\right\|_2}{\|x\|_2} \to 0, \quad x \to 0.$$

(iii) $v(x) \sim O(x^2)$ implies that $\|v(x)\|_2 / \|x\|_2 \to 0$, when $x \to 0$, *i.e.*, $\forall \gamma > 0, \exists r > 0$, such that $\|v(x)\|_2 < \gamma \|x\|_2$ whenever $\|x\|_2 < r$. The inner product between the velocity field $\dot{x}$ and the normal $n_\perp(x) = 2Px$ to the level surface of the Lyapunov function $V(x) = x^T Px$ determines the change of $V(x)$ with time,

$$\begin{aligned}\dot{V}(x) &= (Ax + v(x))^T \cdot 2Px = x^T(A^T P + PA^T)x + 2v^T Px \\ &= -x^T Qx + 2v^T Px.\end{aligned}$$

But $x^T Qx \geq \lambda_{\min}(Q) \|x\|_2^2$, where $\lambda_{\min}(Q) > 0$ is the smallest eigenvalue of the positive definite symmetric matrix Q, and

$$2v^T Px \leq 2\|v\|_2 \|Px\|_2 \leq 2\gamma \|x\|_2 \|P\|_2 \|x\|_2, \quad \forall \|x\|_2 < r. \tag{7}$$

Therefore

$$\begin{aligned}\dot{V}(x) &\leq -\lambda_{\min} \|x\|_2^2 + 2\gamma \|P\|_2 \|x\|_2^2 \\ &= -(\lambda_{\min} - 2\gamma \|P\|_2) \|x\|_2^2.\end{aligned}$$

Choosing $\gamma < \lambda_{\min}(Q)/2\|P\|_2$ ensures that $\dot{V}(x)$ is negative for $x \neq 0$.

(iv) We may further restrict r so that not only the above inequality (7) but also the constraint $\{x : \|x\|_2 < r\} \cup \{x : x^T Px = r\} \subset \mathcal{N}_1$ holds. In $\mathcal{N}_1$, the inverse of the diffeomorphism $y(x) = x + \tilde{h}(x)$ is defined as $x(y) = y + \tilde{k}(y)$. The dynamical equation $\dot{x} = f(x) = Ax + v(x)$ transforms to

$$\dot{y} = \left(I + \frac{\partial \tilde{h}}{\partial x}\right)(Ax + v(x)),$$

and the closed surface Σ defined by $x^T Px = r$ maps to the closed surface Σ' in the y-space defined by

$$V'(y) = (y + \tilde{k}(y))^T P(y + \tilde{k}(y)) = r$$

which has a normal $n'_\perp = 2(I + \frac{\partial \tilde{k}}{\partial y})^T P(y + \tilde{k})$. So, we have

$$\begin{aligned}\dot{V}'(y) &= n'_\perp \cdot \dot{y} = 2(y + \tilde{k})^T P\left(I + \frac{\partial \tilde{k}}{\partial y}\right)\left(I + \frac{\partial \tilde{h}}{\partial x}\right)(Ax + v(x)) \\ &= 2x^T P(Ax + v(x)) = \dot{V}(x) < 0,\end{aligned}$$

since $\left(I + \frac{\partial \tilde{k}}{\partial y}\right)(I + \frac{\partial \tilde{h}}{\partial x}) = I$.

(v) Suppose that $\phi(x_0, t)$ is the flow induced in x-space by $\dot{x} = Ax + v(x)$ and $\phi'(y_0, t)$ is the flow in y-space induced by $\dot{y} = Ay$.

Note that in both spaces any integral curve apart from the origin transversely crosses the closed surface Σ and Σ'. According to Lemma 2.2, for any $x \in \Omega \setminus 0$, there exist unique $x_0(x) \in \Sigma$, $t(x) \in \mathbb{R}$, such that

$$x = \phi(x_0(x), t(x)).$$

Similarly, $\forall y \in \Omega' \setminus 0$, there exist unique $y_0(y) \in \Sigma'$, $t'(y)$ such that

$$y = \phi'(y_0(y), t'(y)).$$

Now, it is possible to build a diffeomorphism between Ω and Ω'. According to Hartman's theorem, this can be done in some neighborhoods of the origin that enclose the surface Σ and Σ'. In the large, for any $x \in \Omega \setminus 0$, let

$$y = a(x) = \phi'(\tilde{y}_0(x), t(x)),$$

where $\tilde{y}_0(x) = x_0(x) + \tilde{h}(x_0(x))$. Inversely, for any $y \in \Omega' \setminus 0$, we may find

$$x = b(y) = \phi(\tilde{x}_0(y), t'(y)),$$

where $\tilde{x}_0(y) = y_0(y) + \tilde{k}(y_0(y))$. As ϕ', ϕ, $\tilde{x}_0$, $\tilde{y}_0$, t, t' are all C^1 functions of their argument according to Lemma 2.1, the map a is a C^1 diffeomorphism. It is now easy to see that the commutation relationship

$$a \circ \phi(x, t) = \phi'(a(x), t) = e^{At} a(x)$$

holds in the whole basin of attraction. Moreover, in the neighborhood $\mathcal{N}_1$ of the origin $x = 0$, $a(x) = x + \tilde{h}$, therefore, $Da(0) = I$. This proves the theorem. □

Remark. The map $h(x)$ that defines the above diffeomorphism could be obtained by solving

$$\frac{dx}{dt} = Ax + v(x),$$
$$\frac{dh}{dt} = Ah - v(x), \tag{8}$$

where $h|_\Sigma = \tilde{h}|_\Sigma$. It is easy to see that $d(x+h)/dt = A \cdot (x+h)$ and the value $h(x)$ for $x \notin \Sigma$ is defined by the flow along the integral curve passing x. In other words, $a(x) = x + h(x)$ satisfies

$$a(\phi(x, t)) = \exp(At) a(x)$$

and thus a is an eigenmapping of the Koopman operator of (1) for the associated eigenmatrix A.

Remark. From the above proof, it is easy to see that if a system is locally linearizable, the linearization could be extended to a larger open region if every orbit in this region has a one to one correspondence with a local orbit. Specifically, if we could initially linearize the stable system in Theorem 2.3 in the neighborhood of a simple closed smooth surface enclosing the equilibrium, the linearization can then be extended to the whole basin of attraction with the equilibrium possibly excluded since no integral curve connects it to the surface. Such a linearization generally does not match the one prescribed by Hartman's theorem and hence $\mathbf{D}a(0) = I$ does not hold. In [22] the authors provide a global aymptotic exponential stability that does not necessarily match the spectrum at the origin, using nontrivial results from differential topology, such as cobordism theory.

Corollary 2.1. *If all the eigenvalues of the matrix A in the above theorem have positive real parts, we may apply Theorem 2.3 to the time reversed system $\dot{x} = -Ax - v(x)$ and conclude that the original system is linearizable by a diffeomorphism in the basin of attraction of the time reversed system.*

Corollary 2.2. *For a general dynamical system $\dot{x} = f(x)$, where $f(x) \in C^2(\mathbb{R}^n)$, Theorem 2.3 and Corollary 2.1 may be applied to the flows in a positively (negatively) invariant region that is homeomorphic to an open set of $\mathbb{R}^m$ with $m < n$ on the stable manifold and the unstable manifold of a stationary point.*

2.3. Linearization of diffeomorphisms

Lemma 2.3 (*Partition of Invariant Set*)**.** *Consider a C^1 homeomorphism $x_{m+1} = f(x_m)$, $m \in \mathbb{Z}$ defined on an open subset D of $\mathbb{R}^n$ that contains the origin $x = 0$ where $f(0) = 0$. Let Σ_- be an open set homeomorphic to a ball B_r and contain the origin with its surface Σ being a C^1 manifold. We define the complement $\Sigma_+ = D \setminus \{\Sigma \cup \Sigma_-\}$. For each integer $m \in \mathbb{Z}$, define $\Sigma^m = f^m(\Sigma)$. Let Σ_+^m and Σ_-^m be defined similarly to Σ_+ and Σ_-. Suppose that for any open neighborhood $\mathcal{N}$ of the origin, $\exists\, m \in \mathbb{Z}$ such that $\Sigma^m \subset \mathcal{N}$. If $D = \cup_{m\in\mathbb{Z}} \Sigma_-^m$ and $\Sigma^1 \cap \Sigma^0 = \emptyset$, then $\forall x \in D$ and $x \neq 0$, $\exists$ unique $m(x) \in \mathbb{Z}$, $x_0(x) \in \Omega_0$ such that $x = f^{m(x)}(x_0(x))$, where $\Omega_0 = \{x : x$ is between Σ and Σ^1, including Σ^1 but not $\Sigma\}$.*

Proof. For each $m \in \mathbb{Z}$, D is partitioned into three disjoint parts, $D = \Sigma_+^m \cup \Sigma^m \cup \Sigma_-^m$ with the origin $0 \in \Sigma_-^m$. As $f(x)$ is a homeomorphism, all Σ^m is homeomorphic to a sphere S^{n-1} and all Σ_-^m is homeomorphic to an open ball B^n. Since $\Sigma^0 \cap \Sigma^1 = \emptyset$ (assumption), without loss of generality, we assume $\Sigma^1 \subset \Sigma_-$. If this is not the case, then $\Sigma \subset \Sigma_-^1$, we may switch the role of Σ and Σ^1 and consider the inverse map f^{-1}.

(1) we claim $\Sigma \cap \Sigma^{-1} = \emptyset$ and $\Sigma_- \subset \Sigma_-^{-1}$.

If $\exists\, x \in \Sigma \cap \Sigma^{-1}$, then $f(x) \in \Sigma^1 \cap \Sigma$, in contradiction to the assumption $\Sigma^1 \cap \Sigma^0 = \emptyset$. Thus, $\Sigma \cap \Sigma^{-1} = \emptyset$.

Take one point $x \in \Sigma^{-1} \subset \Sigma_- \cup \Sigma_+$, since $\Sigma \cap \Sigma^{-1} = \emptyset$. If $x \in \Sigma_-$, take a continuous curve $\gamma \subset \Sigma_-$ connecting x and 0. Then $f(\gamma)$ must intersect $\Sigma^1 \subset \Sigma_-$ since $f(0) = 0$ and $f(x) \in \Sigma$. So, we may pick up a point $y \in \gamma \subset \Sigma_-$ and $f(y) \in \Sigma^1$. But this is impossible since $f^{-1}(\Sigma^1) = \Sigma$ and $y \notin \Sigma$. Therefore, $\Sigma^{-1} \cap \Sigma_- = \emptyset$. We conclude that $\Sigma^{-1} \subset \Sigma_+$ and thus $\Sigma_- \subset \Sigma_-^{-1}$.

Similarly, we can prove $\forall\, m \in \mathbb{Z}$, $\Sigma^m \cap \Sigma^{m-1} = \emptyset$ and $\Sigma_-^m \subset \Sigma_-^{m-1}$. Accordingly, we may define disjoint sets $\Omega_m = \Sigma_-^m \setminus \Sigma_-^{m+1}$, $\forall\, m \in \mathbb{Z}$. Note that this definition is consistent with the definition of Ω_0 in the theorem.

(2) $\forall\, x \in \Omega_m = \Sigma_-^m \setminus \Sigma_-^{m+1}$, we have $f(x) \in \Sigma_-^{m+1} \setminus \Sigma_-^{m+2}$, thus $f(x) \in \Omega_{m+1}$. Vice versa, $\forall x \in \Omega_{m+1}$, then $f^{-1}(x) \in \Omega_m$. Thus Ω_m is homeomorphic to Ω_{m+1} under the homomorphism $f(x)$. Generally, $\Omega_m = f^m(\Omega_0)$.

According to the assumption in the theorem, for any neighborhood $\mathcal{N}$ containing the origin, $\exists\, m \in \mathbb{Z}$, $\Sigma_-^m \subset \mathcal{N}$. So, $\Omega_m \in \mathcal{N}$. Therefore, $D = \cup_{m\in\mathbb{Z}} \Omega_m \cup \{x = 0\}$ since $D = \cup_{m\in\mathbb{Z}} \Sigma_-^m$.

(3) From (1) and (2), we conclude that D is partitioned into disjoint sets $x = 0$ and Ω_m's defined by the homeomorphism $f(x)$. $\forall\, x \neq 0$, $x \in D$, we may find a unique $m \in \mathbb{Z}$ such that $x \in \Omega_m$. Define $m(x) = m$ and $x_0(x) = f^{-m}(x)$, then $x_0(x) \in \Omega_0$ and $x = f^{m(x)}(x_0(x))$. □

Theorem 2.4 (*Linearization of Diffeomorphisms*)**.** *Consider a diffeomorphism $x_{m+1} = f(x_m) = Ax_m + v(x_m)$ defined on $\mathbb{R}^n$, where $v(x) \sim O(|x|^2)$ is C^2 differentiable and A is an $n \times n$ matrix with magnitude of all eigenvalues smaller than 1, then in the basin of attraction D of the origin $x = 0$, there exists a diffeomorphism $y = a(x) = x + h(x)$ with $\mathbf{D}a(0) = I$ which transforms the original map $f(x)$ to a linear one $y_{m+1} = Ay_m$.*

Proof. (1) For matrix A whose eigenvalues have magnitude smaller than 1, there exists a matrix B, such that

$$e^B = A$$

and all the eigenvalues of B have negative real parts, *i.e.* B is a Hurwitz matrix. As mentioned above, for any positive definite symmetric matrix Q, the Lyapunov equation

$$PB + B^T P = -Q$$

has a unique solution P which is positive definite and symmetric as well.

(2) Consider the following dynamical system

$$\dot{z} = Bz,$$

the solution of which could be written as $z(t) = e^{Bt} z_0$. Taking $t = 1$ defines a linear map

$$z_1 = z(1) = e^B z_0 = Az_0.$$

On the other hand, let's estimate

$$\frac{dz^T Pz}{dt} = z^T (PB + B^T P)z = -z^T Qz \le -\alpha z^T Pz,$$

where $\alpha = \lambda_{\min}(Q)/\lambda_{\max}(P) > 0$. According to Gronwall's inequality

$$z(t)^T Pz(t) \le z_0^T Pz_0 e^{-\alpha t}.$$

Taking $t = 1$ results in

$$z_1^T Pz_1 \le z_0^T Pz_0 e^{-\alpha}, \quad \text{or} \quad z_0^T A^T PAz_0 \le z_0^T Pz_0 e^{-\alpha}.$$

(3) Since $\|v(x)\|_2 / \|x\|_2 \to 0$ when $x \to 0$, $\forall \gamma > 0, \exists r > 0$, such that $\|v(x)\|_2 < \gamma \|x\|_2$, for $\|x\|_2 < r$. Now consider

$$\begin{aligned} x_{m+1}^T Px_{m+1} &= (Ax_m + v(x_m))^T P(Ax_m + v(x_m)) \\ &= x_m^T A^T PAx_m + 2x_m^T A^T Pv + v^T Pv \\ &\le x_m^T Px_m e^{-\alpha} + v^T (Pv + 2PAx_m). \end{aligned}$$

But $x_m^T Px_m \ge \lambda_{\min}(P) \|x_m\|_2^2$, and for $\|x_m\|_2 < r$

$$\begin{aligned} v^T (Pv + 2PAx_m) &\le \gamma \|x_m\|_2 (\gamma \|P\|_2 \|x_m\|_2 \\ &\quad + 2 \|P\|_2 \|A\|_2 \|x_m\|_2) \\ &= \gamma \|x_m\|_2^2 \|P\|_2 (\gamma + 2 \|A\|_2). \end{aligned}$$

As a result

$$\begin{aligned} & x_{m+1}^T Px_{m+1} - x_m^T Px_m \\ & \le x_m^T Px_m (e^{-\alpha} - 1) + v^T (PV + 2PAx_m) \\ & \le -(1 - e^{-\alpha})\lambda_{\min}(P) \|x_m\|_2^2 + \gamma \|x_m\|_2^2 \|P\|_2 (\gamma + 2 \|A\|_2) \\ & \le -\|x_m\|_2^2 [(1 - e^{-\alpha})\lambda_{\min}(P) - \gamma \|P\|_2 (\gamma + 2 \|A\|_2)]. \end{aligned}$$

If we choose

$$\gamma < (1 - e^{-\alpha})\lambda_{\min}(P)/ \|P\|_2 (1 + 2 \|A\|_2),$$

then $x_{m+1}^T Px_{m+1} < x_m^T Px_m$.

(4) According to Hartman's Theorem for maps, there exists a C^1 diffeomorphism $y(x) = x + \tilde{h}(x)$ in a neighborhood $\mathcal{N}_1$ of $x = 0$, such that

$$\dot{y} = Ay \quad \text{and} \quad \frac{\left\|\tilde{h}(x)\right\|_2}{\|x\|_2} \to 0, \quad x \to 0.$$

Assume that its inverse is $x(y) = y + \tilde{k}(y)$.

(5) The above r could be chosen to further satisfy

$$\{x : \|x\|_2 < r\} \cup \Sigma \cup \Sigma^1 \subset \mathcal{N}_1,$$

where $\Sigma = \{x : (Ax + v(x))^T P(Ax + v(x)) = r^2\}$ is a simple closed surface enclosing the origin $x = 0$ and $\Sigma^1 = \{x : x^T Px = r^2\}$ is its image under the map $f(x)$.

$\forall z \in \Sigma^1 = f(\Sigma), \exists x \in \Sigma$, such that $z = f(x)$ and according to (3),

$$z^T Pz < x^T Px = r^2, \quad \text{so}, \ \Sigma^1 \cap \Sigma = \emptyset,$$

and $\Sigma_-^1 \subset \Sigma_-$, where we have used the notations in Lemma 2.3. Under the diffeomorphism $y(x) = x + \tilde{h}(x)$ in $\mathcal{N}_1$, Σ corresponds to a surface Σ' in the y-space and Σ^1 to Σ'^1. Also,

$$\Sigma' \cap \Sigma'^1 = \emptyset, \quad \Sigma_-'^1 \subset \Sigma_-'.$$

(6) According to Lemma 2.3, $\forall x \in D, \exists$ unique $x_0(x) \in \Omega_0$ and $m(x) \in \mathbb{Z}$, such that $x = f^{m(x)}(x_0(x))$, where $\Omega_i,\ i \in \mathbb{Z}$ is defined as in Lemma 2.3. Similarly, $\forall y \in D', \exists$ unique $y_0(y) \in \Omega'^0$ and $m'(y)$, such that $y = A^{m'(y)} y_0(y)$. Utilizing $\tilde{h}, \tilde{k}$, it is now possible to build a one-to-one correspondence between D and D'. For any $x \in D$, we have the corresponding $y = a(x) = A^{m(x)}\tilde{y}_0(x)$, where $\tilde{y}_0(x) = x_0(x) + \tilde{h}(x_0(x))$. The inverse map is $x = b(y) = f^{m'(y)}(\tilde{x}_0(y))$, where $\tilde{x}_0(y) = y_0(y) + \tilde{k}(y_0(y))$. $f, \tilde{k}, \tilde{h}, \tilde{x}_0, \tilde{y}_0$ are all C^1-functions of their arguments. For $x \in D \backslash (\cup_{k\in\mathbb{N}} \Sigma^k)$, $y \in D' \backslash (\cup_{k\in\mathbb{N}} \Sigma'^k)$, the functions $m(x), m'(y)$ are also C^1-functions, in this case the maps $b(y), a(x)$ are C^1-differentiable. In another word, the basin of attractions are partitioned into shells which are the open domains between Σ^k and Σ^{k+1} (or between Σ'^k and Σ'^{k+1} in the y-space) and we proved that the above constructed maps are diffeomorphisms within the shells.

However, across the shell boundaries $\{\Sigma^k, \Sigma'^k\}_{k\in\mathbb{N}}$, the functions m, m' jump since they only take discrete integer values. Next, we argue that the maps $b(y), a(x)$ are still C^1-differentiable even at these boundaries. For this purpose, a new partition is taken in both the x- and the y-space such that each of the old boundaries does not intersect the new boundaries. It can be proven that the resulting new map between x and y is identical to the old one based on our constructing procedure if we start with the same Hartman–Grobman map in the local neighborhood of the fixed point. As the old boundaries appear within individual new shells, the maps $a(x), b(y)$ are actually C^1 at these old boundaries according to the conclusion reached above whereas based on the new partition. Therefore, the correspondence maps are C^1-diffeomorphosims and the commutation relation

$$a \circ f^m(x) = A^m a(x)$$

holds in the whole basin of attraction. In $\mathcal{N}_1$, $a(x) = x + \tilde{h}(x)$, we have $\mathbf{D}a(0) = I$. This proves the theorem. □

Remark. The transformation $h(x)$ may be obtained by solving the set

$$\begin{aligned} x_{m+1} &= Ax_m + v(x_m) \\ h_{m+1} &= Ah_m - v(x_m), \end{aligned}$$

with the boundary condition $h|_{x\in\Sigma_-} = \tilde{h}(x)$, since it is easy to see that $(x + h)_{m+1} = A(x + h)_m$.

Remark. A very similar method has been used to prove the Hartman–Grobman theorem in [27]. However, there the linearization map is only proved to be a homeomorphism.

Corollary 2.3. *In Theorem 2.4, if all the eigenvalues of the matrix A have a magnitude greater than 1, we may apply the theorem to the inverse map $f^{-1}(x)$ and obtain the conclusion that $f(x)$ is linearizable in the basin of attraction of $f^{-1}(x)$ through a diffeomorphism $y = a(x)$ with $\mathbf{D}a(0) = I$.*

Corollary 2.4. *Theorem 2.4 and Corollary 2.3 may be applied to a diffeomorphism restricted to a properly chosen region (see Corollary 2.2) on the stable or unstable manifold of a fixed point.*

2.4. Linearization of time-dependent flow

Theorem 2.5 (*Linearization for Periodic Flows*)**.** *Consider a 2π-periodic system*

$$\dot{x} = f(x, t) = A(t)x + v(x, t), \tag{9}$$

where $v(x, t) \sim O(x^2)$ is a C^2-function. If the corresponding linear system $\dot{y} = A(t)y$ is stable, then $x = 0$ is stable and Eq. (9) is *linearizable in its basin of attraction to* $\dot{y} = A(t)y$ *by a* C^1*-diffeomorphism* $y = a(x, t) = x + h(x, t)$ *with* $\mathbf{D}_x a(0, 0) = I$.

Proof. (i) According to Floquet's theory, the solution of the linear system $\dot{y} = A(t)y$ can be written as

$$y(t) = P(t)e^{Bt}y(0),$$

where $P(t) = P(t + 2\pi)$ is a C^1-matrix with $P(0) = I$. B is a constant matrix which determines the stability of the system. As the origin is stable, all eigenvalues of B have negative real parts.

(ii) Define a linear map by the 2π-time evolution of the linear system:

$$y_{m+1} = e^{2\pi B}y_m. \tag{10}$$

The corresponding 2π-period map of Eq. (9) could be written as

$$x_{m+1} = g(x_m) = e^{2\pi B}x_m + w(x_m),$$

where $w(x) \sim O(x^2)$ is a C^2-function. Obviously, the origin $x_m = 0$ is a stable fixed point and its basin of attraction Ω coincides with that of Eq. (9). The map $g(x_m)$ is actually a C^2-diffeomorphism defined on $\mathbb{R}^n$ since it is induced by the ODE flow $\phi(x, t)$. More generally, the flow depends also on the initial time and has to be denoted as $\Phi(x, t, t_0)$. Here we use the simplified notation with the understanding $t_0 = 0$. According to Theorem 2.4, there exists a C^1-diffeomorphism $y(x) = x + \tilde{h}(x)$ defined on Ω, such that $g(x_m)$ is linearized into Eq. (10). We denote the inverse diffeomorphism as $x(y) = y + \tilde{k}(y)$. In the y-space, we denote the flow induced by $\dot{y} = A(t)y$ as $\phi'(y, t)$ and the corresponding basin of attraction of the stable fixed point $y = 0$ is denoted as $\Omega' \subseteq \mathbb{R}^n$.

(iii) Consider the extended phase space $\mathbb{R}^n \times S^1$, where S^1 represents a circle of length 2π parameterizing the time variable. It is possible to build a diffeomorphism between $\Omega \times S^1$ and $\Omega' \times S^1$ as follows.

First, according to (ii), we know that such a diffeomorphism exists between the section $\Sigma_0 = \{(x, 0) : x \in \Omega\}$ and $\Sigma'_0 = \{(y, 0) : y \in \Omega'\}$. More generally,

$\forall (x, t) \in \Omega \times S^1, t \in (-2\pi, 0], \exists$ unique $x_0(x) \in \Sigma_0$, such that, $x = \phi(x_0(x), t)$.

The corresponding point in $\Omega' \times S^1$ is

$$y = a(x, t) \equiv \phi'(\tilde{y}_0(x), t), \quad t' = t,$$

where $\tilde{y}_0(x) = x_0(x) + \tilde{h}(x_0(x))$.

Inversely, $\forall\, (y, t') \in \Omega' \times S^1, t' \in (-2\pi, 0], \exists$ unique $y_0(y') \in \Sigma'_0$, such that $y = \phi'(y_0(y), t')$.

The corresponding point in $\Omega \times S^1$ is

$$x = b(y, t') \equiv \phi(\tilde{x}_0(y), t'), \quad t = t',$$

where $\tilde{x}_0(y) = y_0(y) + \tilde{k}(y_0(y))$. For $t = -2\pi, y = \phi'(y_0, -2\pi) = e^{-2\pi B}y_0$ corresponds to $x = \phi(x_0, -2\pi) = g^{-1}(x_0)$, the same as that determined by the diffeomorphism $y(x) = x + \tilde{h}(x)$. Therefore, in view of the 2π-periodicity, the diffeomorphism $(a(x, t), I(t)) : (x, t) \to (y(x, t), t')$ defined by (see Fig. 1)

$$I(t) = t$$

$$\begin{aligned} y(x, t) &= a(x, t) \equiv P(t)e^{Bt}e^{-m2\pi B}a(g^m(\Phi(x, -t, t))) \\ &= P(t)e^{Bt}a(\Phi(x, -t, t)) = P(t)e^{Bt}y(0), \quad \forall m \in \mathbb{Z} \end{aligned} \tag{11}$$

where $e^{-m2\pi B}a(g^m(\Phi(x, -t, t))) = a(\Phi(x, -t, t))$ holds because g and $e^{2\pi B}$ commute, holds for all time t and coincide with $x + \tilde{h}(x)$ on the section Σ_0. We may write it as

$$a(x, t) = x + h(x, t),$$

where $h(x, t) = h(x, t + 2\pi)$ and $D_x a(0, 0) = I$. □

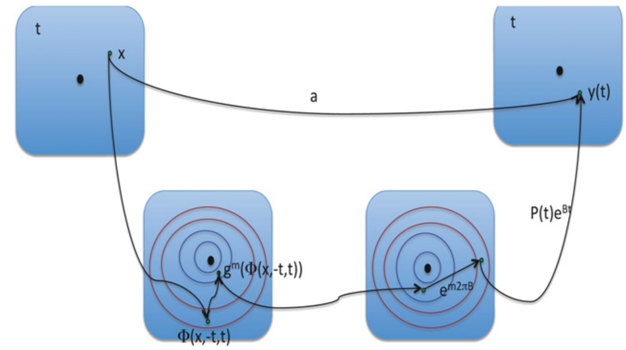

Fig. 1. Extension of the domain of definition of the map $a(x, t)$ to the whole basin of attraction as analytically shown in Eq. (11), based on periodicity and commutativity.

Remark. We can use

$$\dot{x} = A(t)x + v(x, t)$$
$$\dot{h} = A(t)h - v(x, t) \quad \text{with } h|_{(x,0)\in\Sigma_0} = \tilde{h}(x)$$

to determine the transformation $h(x, t)$. From Floquet theory [28] we know that a time-dependent transformation $Py = x$ leads to

$$\dot{y} = By,$$

where B is a time-independent matrix. Let V be such that $V^{-1}BV = \Lambda$, where Λ is a diagonal matrix. Then by setting

$$z = V^{-1}y = V^{-1}P^{-1}x,$$

and suspending the system as $\dot{x} = A(s)x, \dot{s} = 1$, we obtain $z(s, x)$ as Koopman eigenfunctions of the suspended system. These Koopman eigenfunctions are associated with the Floquet exponents.

Corollary 2.5. *The theorem can easily be extended to the unstable equilibrium of a time periodic system.*

Corollary 2.6. *Theorem 2.5 and Corollary 2.5 can be applied to the flow on a properly chosen region (see Corollary 2.2) of the stable or unstable manifold of a stationary point of a time-periodic system.*

2.5. Linearization around a periodic orbit

Theorem 2.6 (*Linearization for Periodic Orbit*)**.** *Suppose that in (1),* $v(x) \in C^2(D)$*, and the system has a stable periodic orbit* γ *with the period* T*. Then, in the basin of attraction* Ω *of* γ*, the system (1) is conjugate to a time-periodic linear flow* $\dot{y} = A(t)y$*, where* $A(t)$ *is the velocity gradient matrix in the transverse direction along* γ *(defined below) with* $A(t + 2\pi) = A(t), y \in \mathbb{R}^{n-1}$.

Proof. Without loss of generality, assume that the period of γ is $T = 2\pi$ (this can always be achieved by a rescaling of time).

(1) The stability matrix J of the periodic orbit γ is obtained through integration of the following equation

$$\frac{dJ}{dt} = EJ$$

along the periodic orbit, where $E = \mathbf{D}f(x)$. Because γ is stable, all eigenvalues of J except the leading one have a magnitude smaller than 1. The leading eigenvalue assumes the value of 1, corresponding to translation along the flow direction. As $v(x) \in C^2(D)$, it is possible to build a C^2-coordinate system in a small neighborhood $\mathcal{N} \subset D$ of γ contained in its basin of attraction,

$$s = s(x), \qquad z = z(x),$$

where $s \in \mathbb{R}$ is a periodic coordinate in the longitudinal direction and $z \in \mathbb{R}^{n-1}$ (the transverse direction coordinate) being zero on γ, such that Eq. (1) transforms to

$$\dot{s} = 1, \qquad \dot{z} = g(z, s), \tag{12}$$

where $s(0) = s(2\pi)$ indicates the 2π period and $g(z, s) = g(z, s + 2\pi)$ has continuous second derivatives [29,30].

(2) In the second equation of Eq. (12), there is no explicit time dependence. However if we identify s with t, a time periodic system

$$\dot{z} = g(z, t) \tag{13}$$

is obtained. In the new coordinate system, $z = 0$ is a stable equilibrium of Eq. (13), its stability inherited from γ. According to Theorem 2.5, Eq. (13) is conjugate to

$$\dot{y} = A(t)y, \tag{14}$$

where $A(t) = Dg|_{z=0}, A(t + 2\pi) = A(t)$ by a C^1-map $y = b(z, t)$. Hence, in $\mathscr{N}$, Eq. (1) is conjugate to Eq. (14) through a diffeomorphism $\tilde{a}(x) : x \to (s(x), z(x)) \to (t(x), b(z(x)))$.

(3) This diffeomorphism can be easily extended to Ω.

Take a section $\Sigma = \{x : x \in \mathscr{N}, s(x) = 0\}$ of $\mathscr{N}$, which corresponds to the section Σ' in the (t, y)-space by the diffeomorphism $\tilde{a}(x)$. $\forall x \in \Omega$, we build a reversible mapping as follows:

if $x \in \mathscr{N}$, then the corresponding point is $(t, y) = \tilde{a}(x)$;

if $x \notin \mathscr{N}$, the orbit passing x will eventually intersect Σ since γ is stable. The corresponding point is then $(-t, y) = (-t, \phi'(y_0, -t))$, where $\phi'(y, t)$ is the flow defined by Eq. (14) and $(0, y_0) = \tilde{a}(x_0)$, x_0 being the first intersection with Σ of the orbit passing x and t being the traveling time from x to the intersection.

We may define the inverse mapping in a similar way. In this manner, the conjugacy is extended to the whole Ω. We write it as $(t, y) = a(x)$. Note that in Eq. (14), we may identify t with $t + 2\pi$ since it is a 2π-periodic system. When this is done $a(x)$ is a diffeomorphism. □

Remark. The conjugacy defined in Theorem 2.6 provides a convenient cyclic coordinate system in the basin of attraction of a periodic orbit. The usual construction of coordinate system along the cycle is only valid locally. The functions $\exp(i2\pi ns(x))$ are Koopman operator eigenfunctions associated with eigenvalues $\exp(i2\pi n)$. Additional Koopman operator eigenfunctions can be obtained by following our Remark after Theorem 2.5, and replacing t with s.

Corollary 2.7. *If γ is unstable, we have similar conjugacy in the basin of attraction for the time-reversed system.*

Corollary 2.8. *Theorem 2.6 and Corollary 2.7 can also be applied to the flow in a properly chosen region (see Corollary 2.2) of the stable or the unstable manifold of a periodic orbit.*

3. Several examples

In this section, we will linearize several most studied ODEs around equilibria or limit cycles. If the exact solution is available, the linearization is also exact in the whole basin of attraction. For systems with no exact solutions available, we have to resort to analytical approximation or numerical integration to produce an approximate linearization.

As in general we do not know the explicit functional form of the local linearization prescribed by the Hartman Theorem, the previous theorems could not be directly applied. Based on the Remark after Theorem 2.3, an alternative procedure is designed which identifies points on a closed curve in the original and the linearized phase space and then determines the linearization map in the whole basin of attraction according to the flows in the two spaces. So, the linearization maps obtained may not match exactly Hartman's prescription. Especially, the result $\mathbf{D}a = I$ does not hold at the equilibrium but does hold on the closed curve we selected.

3.1. One-dimensional model

Let's consider the following one-dimensional system defined on the real line,

$$\dot{x} = x - x^3, \tag{15}$$

which has two stable equilibria $x = \pm 1$ and one unstable point $x = 0$. The solution of Eq. (15) can be written explicitly as

$$x(t) = \frac{x_0}{\sqrt{(1 - x_0^2)e^{-2(t-t_0)} + x_0^2}}, \tag{16}$$

where $x_0 = x(t_0)$. According to Theorem 2.3, and Corollary 2.1, we may derive three qualitatively different linearizations around these equilibria.

(1) Linearization around $x = 0$: the linearized dynamics is $\dot{y} = y$ and its solution is $y = y_0 \exp(t - t_0)$. If x is identified with y near the origin, *i.e.*,

$$r = x_0 = y_0 = ye^{-(t-t_0)}$$

for a given $0 < r^2 \ll 1$, a mapping between the repelling region $\{x : x \in (-1, 1)\}$ and the y-space can be established:

$$x = b(y) = \frac{y}{\sqrt{1 + y^2 - r^2}}, \tag{17}$$

for all $y \in \mathbb{R}$. The mapping is not unique since for different r, we have different maps. Note that $db(y)/dy|_{y=0} = 1$ does not hold as we explained previously but becomes a better and better approximation when r^2 is decreasing. In fact, it is possible to set $r = 0$ in Eq. (17) which then becomes the map implied in the Hartman–Grobman theorem. In that case, the linearizing transformation $y = x/(1 - x^2)^{1/2}$ is the Koopman eigenfunction with eigenvalue 1, as is easily checked by verifying $\dot{y} = y$. With $y \to \pm\infty, x \to \pm 1$. The inverse map could be obtained by solving Eq. (17) for y in terms of x or use Eq. (8).

(2) Linearization around $x = 1$: if we make the choice that the linearized dynamics has the form $\dot{y} = -2(y - 1)$ and identify x with y near $x = y = 1$, *i.e.*,

$$r = x_0 - 1 = y_0 - 1 = (y - 1)e^{-2(t-t_0)},$$

where $r^2 \ll 1$, the linearization map could be written as

$$x = b(y) = \frac{1 + r}{\sqrt{(1 + r)^2 - (y - 1)(r + 2)}}, \tag{18}$$

which has the property that $x \to 0$ when $y \to -\infty$ and $x \to \infty$ when $y \to 1 + (1 + r)^2/(r + 2)$. $y = 1$ always corresponds to $x = 1$. What is interesting in Eq. (18) is that the map is not defined for all $y \in \mathbb{R}$ but terminated at $y = 1 + (1 + r)^2/(r + 2)$, which is ascribed to the blowup of the solution of Eq. (15) in the $x > 1$ region in finite time while the exponential function prescribed by the linearized equation is always finite for finite times.

(3) Around $x = -1$, the linearization transformation is similar

$$x = b(y) = \frac{-1 + r}{\sqrt{(-1 + r)^2 + (y + 1)(2 - r)}}, \tag{19}$$

which approaches $-\infty$ at $y = -1 - (-1 + r)^2/(2 - r)$, due to the reason as explained above.

3.2. A 2-dimensional flow with one unstable equilibrium

This example is taken from the textbook by Lawrence Perko [31]

$$\dot{z}_1 = 2z_1$$

$$\dot{z}_2 = 4z_2 + z_1^2, \tag{20}$$

which can be solved exactly

$$z_1 = z_{10}e^{2t}$$
$$z_2 = z_{20}e^{4t} + z_{10}^2 te^{4t}, \tag{21}$$

where z_{10}, z_{20} are initial positions. Note that there is a secular term in the second equation of (21), which brings complication to the linearization of Eq. (20) into the target linear system $\dot{y}_1 = 2y_1, \dot{y}_2 = 4y_2$ which has a general solution $y_1 = y_{10}e^{2t}, y_2 = y_{20}e^{4t}$. Just as in the previous example, in order to build up the linearization map we may impose $(y_{10}, y_{20}) = (z_{10}, z_{20})$ on the small circles $z_{10}^2 + z_{20}^2 = r^2$ and $y_{10}^2 + y_{20}^2 = r^2$ with $r \ll 1$. From these, we may compute the time needed to reach (y_1, y_2) from the small circle in y-space.

$$t(y_1, y_2) = \frac{1}{4}\ln\left(\frac{1}{2}\left(\frac{y_1}{r}\right)^2 + \frac{1}{2}\sqrt{\left(\frac{y_1}{r}\right)^4 + 4\left(\frac{y_2}{r}\right)^2}\right), \tag{22}$$

and the linearization map is

$$z_1 = y_1$$
$$z_2 = y_2 + t(y_1, y_2)y_1^2. \tag{23}$$

Note that the map is not analytic at the origin because of the logarithmic function in $t(y_1, y_2)$. One interesting observation is that the function $t(y_1, y_2)$ can be chosen as

$$t(y_1, y_2) = \frac{1}{4}\ln y_1^2, \tag{24}$$

so that Eq. (23) becomes a C^1 transformation dictated by the Hartman–Grobman theorem. Eq. (24) can be obtained from (22) by taking r to zero and neglecting the constant but divergent term $\ln(1/r^2)$. It is again interesting to note the connection with Koopman operator eigenfunctions: $f_1 = y_1$ and $f_2 = y_2 + y_1^2 \ln y_1^2/4$ are Koopman eigenfunctions at eigenvalues 2 and 4, respectively, as is easily verifiable by a direct computation. In fact, Eq. (20) could not be linearized by an entire function in the neighborhood of the origin as suggested by the normal form theory [14,32] since a resonance term exists in the second equation of Eq. (20). However, our scheme shows that it is possible to linearize it with a function that is C^1.

3.3. Rayleigh equation

The Rayleigh equation describes nonlinear systems with one degree of freedom which admit self-sustained oscillations, introduced by Rayleigh in the acoustics study. It finds applications in a wide variety of natural and engineering systems. The Rayleigh equation is

$$\frac{d^2y}{dt^2} + y = \epsilon\left(\frac{dy}{dt} - \frac{1}{3}\left(\frac{dy}{dt}\right)^3\right), \tag{25}$$

where $\epsilon > 0$ describes the amplitude of the linear dissipation and nonlinear agitation. It can be written as a two-component dynamical system

$$\dot{y} = x$$
$$\dot{x} = -y + \epsilon\left(x - \frac{1}{3}x^3\right). \tag{26}$$

Based on the dominant linear part, we make the following transformation

$$y = A\sin(t + \phi)$$
$$x = A\cos(t + \phi), \tag{27}$$

which renders the following set of equations

$$\frac{d\phi}{dt} = -\frac{\epsilon}{2}\left[\left(1 - \frac{A^2}{6}\right)\sin(2t + 2\phi) - \frac{1}{12}A^2\sin(4t + 4\phi)\right]$$
$$\frac{dA}{dt} = \frac{\epsilon A}{2}\left[1 - \frac{A^2}{4} + \left(1 - \frac{A^2}{3}\right)\cos(2t + 2\phi) - \frac{A^2}{12}\cos(4t + 4\phi)\right]. \tag{28}$$

So, ϕ and A change slowly with time, reflecting the smallness of the ϵ-perturbation. After averaging with respect to the fast time [14,33], we obtain for the averaged variable $\bar{\phi}$ and $\bar{A}$

$$\frac{d\bar{\phi}}{dt} = 0$$
$$\frac{d\bar{A}}{dt} = \frac{\epsilon\bar{A}}{2}\left(1 - \frac{\bar{A}^2}{4}\right), \tag{29}$$

which can be conveniently solved as

$$\bar{\phi} = \phi_0$$
$$\bar{A} = A_0\Big/\sqrt{e^{-\epsilon t} + \frac{1}{4}A_0^2(1 - e^{-\epsilon t})}, \tag{30}$$

where $A_0 = \sqrt{x_0^2 + y_0^2}$ and $\phi_0 = \tan^{-1} y_0/x_0$ are the initial phase and amplitude, respectively. When $t \to \infty, \bar{A} \to 2$. To the order of ϵ, Eq. (28) has a general solution

$$\phi = \phi_0 + \frac{\epsilon}{4}\left[\left(1 - \frac{\bar{A}^2}{6}\right)(\cos(2t + 2\phi_0) - \cos 2\phi_0) - \frac{\bar{A}^2}{24}(\cos(4t + 4\phi_0) - \cos 4\phi_0)\right]$$
$$A = \bar{A} + \frac{\epsilon\bar{A}}{4}\left[\left(1 - \frac{\bar{A}^2}{3}\right)(\sin(2t + 2\phi_0) - \sin 2\phi_0) - \frac{\bar{A}^2}{24}(\sin(4t + 4\phi_0) - \sin 4\phi_0)\right]. \tag{31}$$

So, in view of Eq. (27), the asymptotic trajectory is a limit cycle of radius around 2 for small ϵ. Below, we provide an approximation to the linearization conjugacy for Eq. (26) around the origin and around the limit cycle.

(I) The linearized equation around the origin is

$$\dot{y} = x$$
$$\dot{x} = -y + \epsilon x, \tag{32}$$

which, to the order of ϵ, has a general solution

$$x = e^{\epsilon t/2}\left(x_0\cos t - \left(y_0 - \frac{\epsilon}{2}x_0\right)\sin t\right)$$
$$y = e^{\epsilon t/2}\left(y_0\cos t + \left(x_0 - \frac{\epsilon}{2}y_0\right)\sin t\right), \tag{33}$$

where (x_0, y_0) is the initial position. The inverse transformation is obtained from Eq. (33) by making the substitution $t \to -t, x \leftrightarrow x_0, y \leftrightarrow y_0$. Along the solution curve of Eq. (32), the function $I = x^2 + y^2 - \epsilon xy$ satisfies $dI/dt = \epsilon I$ which has a solution $I(t) = I_0\exp(\epsilon t)$. If we identify the points in the phase spaces of Eq. (26) and of Eq. (32) on the small ellipse defined by

$$x_0^2 + y_0^2 - \epsilon x_0 y_0 = \delta^2, \tag{34}$$

where $\delta \ll 1$ indicates the size of the ellipse, then to a general

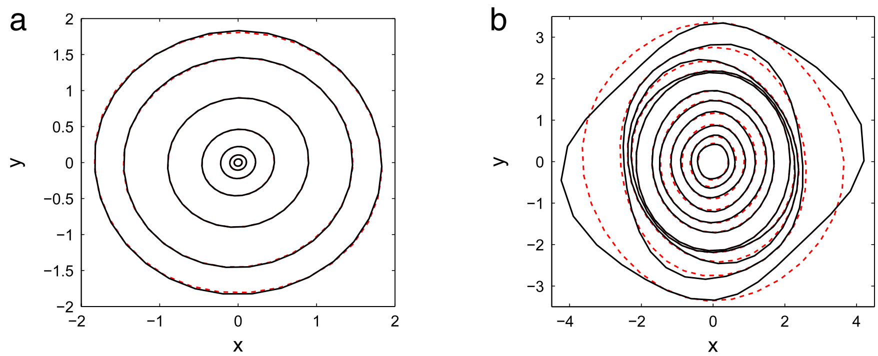

Fig. 2. Comparison of the evolution of closed smooth curves in the neighborhood of the equilibrium (a) and of the stable limit cycle (b), by directly evolving the Rayleigh equation (26) (solid lines) or mapping the trajectories of its linearizations (using approximations described in the text) Eqs. (32) and (37), Eq. (39) (dashed lines).

point (x, y) in the phase space of Eq. (32), there corresponds a point (x_0, y_0) on the curve (34) which takes time

$$t = \frac{1}{\epsilon} \ln \frac{x^2 + y^2 - \epsilon xy}{\delta^2} \tag{35}$$

to reach (x, y). According to Eqs. (30) and (33), for an arbitrary point (x, y) in the phase space of Eq. (32), after determining (x_0, y_0), we can identify the corresponding $\bar{A}$, ϕ_0 in the phase space of Eq. (26) as follows

$$\bar{A}^2 = \frac{x^2 + y^2 - \epsilon xy + \epsilon xy \cos 2t + \frac{\epsilon}{2}(y^2 - x^2) \sin 2t}{1 + \frac{1}{4}(1 - e^{-\epsilon t})\left(x^2 + y^2 - \epsilon xy + \epsilon xy \cos 2t + \frac{\epsilon}{2}(y^2 - x^2) \sin 2t\right)}$$
$$\tan \phi_0 = \frac{y \cos t + \left(\frac{\epsilon}{2} y - x\right) \sin t}{x \cos t + \left(y - \frac{\epsilon}{2} x\right) \sin t}, \tag{36}$$

where t is given by Eq. (35). Using Eq. (36) together with Eq. (31) and Eq. (27), we may locate the point in the phase space of Eq. (26) that corresponds to the point (x, y) in the phase space of Eq. (32). This map is shown in Fig. 2(a) with parameter $\epsilon = 0.1$. We have identified the ellipses defined by Eq. (34) in the phase space of the Rayleigh equation (26) with a similar ellipse in the phase space of its linearization. The initial ellipse is evolved and plotted for six consecutive steps with time interval $t = 15$ between plots. As seen in the Fig. 2(a), the curves corresponding to the original Rayleigh equation (solid lines) match extremely well with those derived from the approximate linearization conjugacy (dashed lines).

(II) Next, we pursue the approximate linearization around the stable limit cycle. In view of Eqs. (27), (30) and (31), $\bar{\phi}$, $\bar{A}$ could be conveniently identified as a set of generalized coordinates around the limit cycle where $\bar{\phi}$ describes the longitudinal motion and $\bar{A}$ the transversal. Linearization of the $\bar{A}$ equation in Eq. (29) leads to

$$\dot{B} = -\epsilon(B - 2), \tag{37}$$

which has a solution

$$B = 2 + (B_0 - 2)e^{-\epsilon t}.$$

Just as in the first example, the map

$$\bar{A} = \frac{2(2 + \delta)}{\sqrt{(4 + \delta)(2 - B) + (2 + \delta)^2}}, \tag{38}$$

where we set $A_0 - 2 = B_0 - 2 = \delta$ with $|\delta| \ll 1$, will linearize the $\bar{A}$ equation and maps a point in the phase space of Eq. (37) to its conjugate. Note that when $B = 2$, $A = 2$ and when $B \to -\infty$, $A \to 0$. Also, the map is only valid for $B < 2 + (2 + \delta)^2/(4 + \delta)$ for the similar reason as explained in the first example. Together with the cyclic coordinate equation

$$\dot{s} = 1, \quad \text{with } s \in S^1. \tag{39}$$

Eq. (37) is a linearized version of Eq. (29) and thus of Eq. (26), with the help of the transformation equations (27) and (31). The mapping in the angular direction is simply

$$\bar{\phi} = \phi_0 = s - \frac{1}{\epsilon} \ln \frac{\delta}{B - 2}, \tag{40}$$

where the second term on the right hand side prescribes the time needed to reach B from the initial δ-circle. If we treat (B, s) as the polar coordinates of certain plane P_L, Eqs. (38) and (40) give the coordinate transformation from (B, s) to $(\bar{A}, \bar{\phi})$, and then to (x, y) by Eqs. (27) and (31). It is convenient to visualize the mapping as shown in Fig. 2(b), which is constructed similar to Fig. 2(a) but with the initial curve in the neighborhood of the limit cycle. The time interval between neighboring curves is $t = -8$ inside the cycle and $t = -4$ outside. The inside parts of two evolutions continue to match well while large deviations can be observed on the outside parts, where the nonlinear terms in Eq. (26) quickly increase which soon renders our approximation solution invalid.

3.4. Duffing equation with spiral centers

In this section, we consider the Duffing equation

$$\ddot{x} + \alpha \dot{x} - \beta x + x^3 = 0, \tag{41}$$

which takes into account the friction and the higher-order non-harmonic force in a mechanical system. In a state space representation, it can be rewritten as

$$\dot{x} = y$$
$$\dot{y} = \beta x - x^3 - \alpha y. \tag{42}$$

For $\beta > 0$, there are three equilibria: $(\pm\sqrt{\beta}, 0)$, $(0, 0)$. If in addition $\alpha > 0$, the first two are two sinks and the origin is a saddle. We could take a similar approach as in the previous example and obtain approximate analytical conjugacy to the linearized system. However, here a different approach is introduced which combines analytical and numerical computation in a unified scheme and is easier to apply to more complex systems. Keeping in mind the theorems proved in earlier sections, we choose a basis set of functions to approximate the linearization mapping around the equilibrium in an extended region. The projection of the mapping on each basis function is determined by minimizing the difference between trajectories of the original nonlinear system and those of their linearizations.

We implement the scheme in part of the basin of attraction of the equilibrium $(\sqrt{\beta}, 0)$, around which the linearized dynamical equations are

$$\dot{u} = v$$
$$\dot{v} = -2\beta u - \alpha v. \tag{43}$$

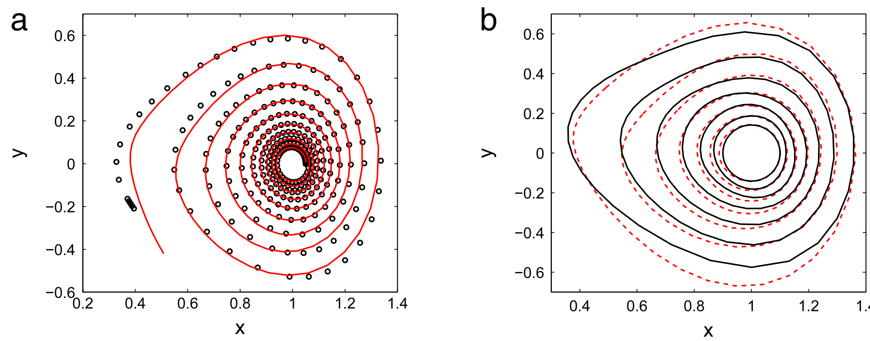

Fig. 3. Comparison of one typical orbit (a) and the evolution of a circle (b) in the neighborhood of the equilibrium ($\sqrt{\beta}$, 0), by directly evolving the Duffing equation (42) (black) or mapping the trajectory of its linearization Eq. (43) (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To be specific, below we set $\beta = 1$, $\alpha = 0.1$. For simplicity, we use a second order map

$$\mathbf{f}(u, v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix} \quad (44)$$

to approximate the linearization conjugacy. The unknown parameters could be determined by the normal form analysis [32] which is only valid locally. Here, based on the previously proved theorems, the linearization can be extended to a much larger region by a different approximation which is to be detailed.

Note that the function

$$I(u, v) = u^2 + \frac{1}{2\beta} v^2 + \frac{\alpha}{2\beta} uv \quad (45)$$

decreases exponentially along the orbit of Eq. (43). As in previous examples, we may identify with each other the two ellipses

$$I(u_0, v_0) = I(x_0 - 1, y_0) = \delta_0,$$

where $\delta_0 \ll 1$. After integrating Eq. (43) from (u_0, v_0) and Eq. (42) from (x_0, y_0) for a finite time T and utilizing the linearization map Eq. (44), we construct an error function

$$\mathrm{Err}(u_0, v_0, T) = \sum_{i=1}^{N} ((x(t_i), y(t_i))^T - \mathbf{f}(u(t_i), v(t_i)))^2, \quad (46)$$

where N is the total number of integration steps and t_i's are integration time points with $t_N = T$. The minimization of $\mathrm{Err}(u_0, v_0, T)$ determines the values of a_i's which depend on the starting position and the integration time. It is not hard to write down a similar error function with multiple orbits. However, the direct use of ansatz equation (44) highly constrains the validity of the linearization region. In depth study reveals that the error mainly originates from the dynamical phase of the orbits. The equilibrium under investigation is a stable spiral. The radial expansion is slow because of the smallness of α but the angular rotation is of the order 1. Therefore, the nonlinear term in Eq. (42) affects mainly the phase of the rotation, which is similar to the secular effect seen in many nonlinear systems. Furthermore, this influence is non-uniform in the phase space so that simple quadratic maps are not able to encode such a phenomenon.

To incorporate this observation, we introduce a phase slip function

$$t_a(u, v) = a_7 + a_8 u + a_9 v + a_{10} u^2 + a_{11} uv + a_{12} v^2, \quad (47)$$

which makes up for the phase shift due to nonlinearity, and the configuration adjustment function

$$\mathbf{g}(u, v) = \begin{pmatrix} a_{13} \cos t_a & -a_{14} \sin t_a \\ a_{15} \sin t_a & a_{16} \cos t_a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (48)$$

which revolves orbits to the correct phase and adjusts their shapes. Combining Eqs. (44), (47) and (48) results in the final approximate linearization map

$$\mathbf{h}(u, v) = \mathbf{f} \circ \mathbf{g}(u, v), \quad (49)$$

with the ansatz equation (49), we used two orbits with initial condition $(u_0, v_0) = (0.1, 0)$ and $(u_0, v_0) = (0, 0.142)$ in the phase space of Eq. (43) (thus two orbits for Eq. (42) with the corresponding initial conditions determined by Eq. (45)) to construct the error function, the minimization of which provides the values of a_i's.

To check the validity of the linearization, we start from (0.05, 0) and integrate Eq. (43) for a duration $t = -45$ and the image of the orbit under $\mathbf{h}(u, v)$ is displayed as solid lines in Fig. 3(a), which agrees well with the corresponding orbit of the Duffing equation shown as circles. The orbit starts to deviate when approaching the boundary of the basin of attraction. Fig. 3(b) displays the evolution of an ellipse under the action of the Duffing equation (solid lines) and of its linearization mapped by Eq. (49) (dashed lines). The time gap between neighboring curves is $t = -5$. The agreement is very good in the close neighborhood of the equilibrium and towards the attraction boundary the shape and time dependence on the phase space location of the Duffing orbit becomes very non-uniform which could not be captured accurately by simple analytical ansatz such as $\mathbf{h}(u, v)$.

3.5. Lorenz equation—linearization on the unstable manifold

The Lorenz equation is a paradigm for nonlinear studies and has the form

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z, \end{aligned} \quad (50)$$

where σ, ρ are the Prandtl number and the Rayleigh number respectively, derived from the fluid convection equations. When $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, Lorenz found that solution of Eq. (50) is neither stationary nor periodic with an attractor that is a compact set of non-integer dimension [34]. Below, we assume that these parameter values are taken.

Eq. (50) has three equilibria

$$\begin{aligned} E_\pm &= (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1), \\ E_0 &= (0, 0, 0). \end{aligned}$$

We choose to linearize Eq. (50) around E_+ and the linearized equation is

$$\dot{x} = -\sigma x + \sigma y$$

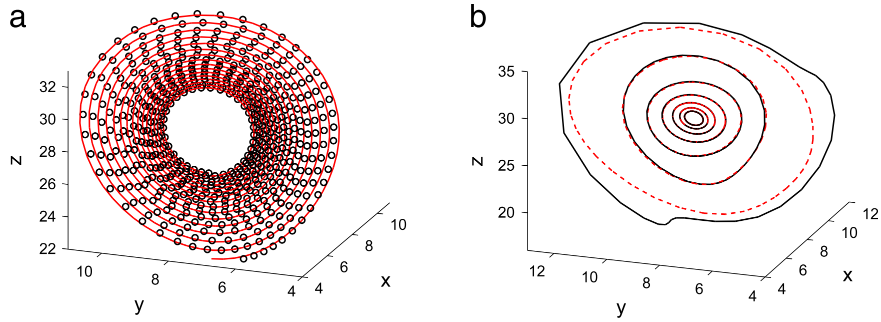

Fig. 4. Comparison of one typical orbit (a) and the evolution of a circle (b) on the unstable manifold of the equilibrium E_+, by directly evolving the Lorenz equation (50) (black) or mapping the trajectory of its linearization Eq. (52) (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\dot{y} = x - y - \sqrt{\beta(\rho-1)}z$$
$$\dot{z} = \sqrt{\beta(\rho-1)}(x+y) - \beta z. \quad (51)$$

The eigenvalues of the corresponding stability matrix include an unstable complex pair and a stable one: $\lambda_\pm = 0.0940 \pm i10.1945$, $\lambda_0 = -13.9546$. The corresponding eigenvector is

$$v_\pm = (-0.2661 \mp 0.2950i, 0.0321 \mp 0.5691i, -0.7192)^t,$$
$$v_0 = (0.8557, -0.3298, -0.3988)^t.$$

So E_+ is a saddle with a 2-d unstable manifold and a 1-d stable manifold. As stated previously, our linearization scheme can be applied to the vector field on the unstable manifold. Similar strategy as employed in the previous example will be adopted. The restriction of Eq. (51) on its unstable plane can be described by a pair of equations

$$\dot{u} = 0.0940\,u - 11.3272\,v$$
$$\dot{v} = 9.1751\,u + 0.0940\,v. \quad (52)$$

As before, the second order mapping

$$\mathbf{f}_l(u,v) = E_+ + \begin{pmatrix} -0.3593 & 0.4425 \\ 0.0434 & 0.8536 \\ -0.9709 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}\begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}$$

is chosen for simplicity, where the linear part maps an orbit of Eq. (52) to that of Eq. (51). In order to extend the validity region, we may define the phase slip function similar to Eq. (47) and the configuration adjustment function similar to Eq. (48). The combined function

$$\mathbf{h}_l(u,v) = \mathbf{f}_l \circ \mathbf{g}(u,v), \quad (53)$$

makes the target approximate linearization map, where 19 unknown parameters $\{a_i\}_{i=1,\dots,19}$ are determined by the error function minimization as suggested in the previous example.

Based on the two orbit segments starting from $(u_0, v_0) = (0.5, 0)$ and $(u_0, v_0) = (-0.5, 0)$ and lasting for $t = 31$, we constructed and minimized the error function, and thus determined parameters $\{a_i\}_{i=1,\dots,19}$. In Fig. 4(a), the image by $\mathbf{h}_l(u,v)$ of the linearized orbit starting with $(u_0, v_0) = (2, 0)$ and evolving for $t = 30$ is depicted with solid line and compared with the corresponding orbit of the Lorenz equation depicted with circles. The agreement is extremely good near the equilibrium and deteriorate toward the brim. Still, the discrepancy occurs mainly due to the secular effects in phase. In Fig. 4(b), we displayed the evolution of a small circle in the Lorenz system (solid line) and in Eq. (52) mapped by $\mathbf{h}_l(u,v)$ (dashed line). The time gap between neighboring curves is $t = 4.3$. Just as before, the discrepancy is perceivable when the curves drift away from E_+, where they start to deform in an intricate way under the influence of the nonlinear terms.

4. Summary

In this paper, we extended the local linearization of the Hartman–Grobman theorem to the whole basin of attraction of a stable equilibrium or limit cycle and connected these results to the spectral theory of Koopman operators. The linearization can be applied to both maps and flows. We also used several most commonly encountered examples to illustrate application of the theory in different cases. Our results show that the orbit structure of a nonlinear system in the basin of attraction of a stable equilibrium or a limit cycle is similar to that of a linear system. If the attractor of a nonlinear system consists only of discrete stable equilibria or limit cycles, then the whole phase space can be partitioned into invariant patches, each being the basin of attraction of an equilibrium or a limit cycle, such that the dynamics in each patch is conjugate to a linear one.

The linearization map predicted by our theorem is multivariate and hence is hard to represent numerically except for one-dimensional case. As shown in Section 3, if we resort to analytical or semi-analytical approximation, we have to either solve the ODEs exactly or try to find a good basis function to approximate the linearization. Although our simple choices based on numerical observation much extended the linearization region, near the boundary of basin of attraction, they go awry quickly. An effective way to approximate the linearization map systematically and in a more automatic manner would be computation of the associated Koopman eigenfunctions [7,4,35].

The theorems proved in this paper only work for stable equilibria and limit cycles, or on appropriately chosen regions of the stable or unstable manifold of saddles. It should be also possible to extend the result to a sizable neighborhood of saddles since Hartman–Grobman theorem is also valid for saddles. The extended neighborhood is surely not the whole phase space in general since for example a chaotic system cannot be conjugate to a linear system. As mentioned in Section 1, a nonlinear flow or map is linearizable in the neighborhood of a normally hyperbolic invariant manifold, it does not seem hard to extend the current result to enlarge the validity domain of this type of linearization.

Acknowledgments

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