ME/CS 132(b)

Notes on the kinematics of Inertial Navigation

1 Introduction

These notes review the basic kinematic equations governing inertial navigation, when the inertial measurement suite consists of an IMU (accelerometers to measure Cartesian motion, and gyroscopes to measure rotational motion).

2 Spherical Kinematics and Rigid Body Rotations

Motions of a 3-dimensional rigid body where one point of the body remains fixed are termed *spherical motions.* A *spherical displacement* is a rigid body displacement where there is a fixed point in the initial and final positions of the moving body. It can be shown that spherical displacements and spherical motions have 3 independent degrees of freedom.

Consider a 3-dimensional rigid body, $\mathcal{B}$. Let C denote the point in $\mathcal{B}$ which is fixed during subsequent spherical motions. Choose a fixed reference frame, $\mathcal{F}$ whose origin lies at point C. To understand what happens to a rigid body during a spherical motion, select an arbitrary point, P_0, in $\mathcal{B}$. Let $\vec{v}_0$ denote the vector from the origin of this fixed reference frame to P_0. I.e., $\vec{v}_0$ denotes the coordinates of P_0 as seen by an observer in frame $\mathcal{F}$. Let the body undergo a spherical displacement. After the displacement, let the new location of the point P_0 be denoted by P_1. Similarly, let $\vec{v}_1$ denote the coordinates of P_1, as seen in frame $\mathcal{F}$.

Let us assume that the transformation of the points in $\mathcal{B}$ during the spherical displacement can be represented by the action of a 3×3 matrix, A, acting on the vector of particle coordinates:

$$\vec{v}_1 = A\vec{v}_0 \ .$$

Because the body is rigid, the distance between the fixed point of the motion (which is also the origin of the fixed reference frame) and the given particle is constant. That is:

$$|\vec{v}_1| = |\vec{v}_0| \ .$$

This implies that:

$$|\vec{v}_1|^2 = |\vec{v}_0|^2 \ \Rightarrow \ \vec{v}_1^T\vec{v}_1 = \vec{v}_0^T\vec{v}_0 \ \Rightarrow \ \vec{v}_0^T A^T A\vec{v}_0 = \vec{v}_0^T\vec{v}_0.$$

Since this relationship holds for any choice of P_0, it must be true that $A^TA = I$. That is, A is an orthogonal matrix.

Recall that for matrices A and B, $det(A^T) = det(A)$ and $det(AB) = det(A)\ det(B)$. Hence, for an orthogonal matrix A, $det(A^TA) = det(A^T)det(A) = det^2(A) = det(I) = 1$. Therefore,

$det(A) = \pm 1$ for any orthogonal matrix A. Those orthogonal matrices whose determinant is $+1$ are physically associated with *rotations*, while those whose determinant is -1 are associated with *reflections*. We will only be concerned with rotations. Note that the set of all 3-dimensional rotations forms a group, whose symbol is $SO(3)$.

2.1 Rodriquez' equation and the Exponential Map

The 3×3 rotation matrix $R \in SO(3)$ contains nine entries, but only 3 of these entries are independent. Hence, we need to find suitable parametrizations of $SO(3)$ based on 3 variables. *Angle-Axis* coordinates are one choice. These coordinates are based on Euler's Theorem, which states that "every rigid body rotations is equivalent to a *simple* rotation about a fixed axis." The single scalar ϕ which quantifies the amount of rotation is one coordinate, and the 3×1 unit vector $\vec{\omega}$ represents the other two coordinates (since only 2 of the 3 components of $\vec{\omega}$ are independent, given that it is a unit length vector). The rotation matrix corresponding to rotation ϕ about rotation axis $\vec{\omega}$ can be computed from *Rodriguez' formula:*

$$R(\phi, \vec{\omega}) \;=\; I \;+\; (\sin\phi)\hat{\omega} \;+\; (1 - \cos\phi)\hat{\omega}^2.$$

where $\hat{\omega}$ is a 3×3 skey symmetric matrix having the form:

$$\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{1}$$

where $\vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$ is the unit vector representing the axis of rotation.

Equivalently, one can show that

$$R(\phi, \vec{\omega}) \;=\; e^{\phi\hat{\omega}}$$

where e^M is the *matrix* exponential of a matrix M. The matrix exponential is defined using a series expansion having the same form as the scalar exponential:

$$e^{\phi\hat{\omega}} \;=\; I + \frac{\phi}{1!}\hat{\omega} + \frac{\phi^2}{2!}\hat{\omega}^2 + \frac{\phi^3}{3!}\hat{\omega}^3 + \cdots\ .$$

Noting that $\hat{\omega}^3 = -\hat{\omega}$, one can show that the matrix exponential simplifies to Rogdriguez' equation.

2.2 Angular Velocity

Spatial Angular Velocity. Let P denote a particular particle in the body of a rotationally moving body. Let ${}^B\vec{P}$ denote the position of the particle relative to an observer who is fixed in the moving body. Let ${}^W\vec{P}$ denote the coordinates of this same particle, as seen by an observer in a fixed frame. The velocity of this particle, as seen by the fixed observer, is:

$$
\begin{aligned}
{}^{W}\vec{V}_P &= \frac{d}{dt}{}^{W}\vec{P}(t) = \frac{d}{dt}(R_{WB}(t)^{B}\vec{P}) = \dot{R}^{B}_{WB}\vec{P} = \dot{R}_{WB}R^{T}_{WB}R^{B}_{WB}\vec{P} \qquad (2)\\
&= (\dot{R}_{WB}R^{T}_{WB})^{W}\vec{P} \triangleq \hat{\omega}^{s}_{WB}\ {}^{W}\vec{P} \qquad (3)
\end{aligned}
$$

where $\vec{\omega}^{s}_{WB}$ is termed the *spatial angular velocity* of the spinning body. The spatial angular velocity is a 3×3 skew symmetric matrix Ω^{s}_{WB}

$$
\begin{bmatrix} 0 & -\Omega^{s}_{z} & \Omega^{s}_{y} \\ \Omega^{s}_{z} & 0 & -\Omega^{s}_{x} \\ -\Omega^{s}_{y} & \Omega^{s}_{x} & 0 \end{bmatrix}
$$

This matrix is isomorphic with a 3×1 vector, $\vec{\Omega}^{s}_{WB}$ that we will also call (by abuse of notation) the spatial angular velocity. At each instant, t, one can think of the rotating body as rotating around the *instantaneous axis of rotation*

$$
\vec{\omega}^{s}_{WB} = ||\vec{\Omega}^{s}_{WB}||^{-1}\ \vec{\Omega}^{s}_{WB}\ .
$$

with a rate of rotation (or angular velocity)

$$
\dot{\phi} = sign(\vec{\Omega})||\vec{\Omega}^{s}_{WB}||.
$$

Body Angular Velocity. By definition, we say that the *body angular velocity* is obtained by transforming the spatial angular velocity to the body frame:

$$
\vec{\Omega}^{b}_{WB} = R^{T}_{WB}\ \vec{\Omega}^{s}_{WB}\ .
$$

3 Inertial Navigation

This section first reviews how to estimate the orientation of an IMU's reference frame from gyroscope measurements, and then addresses the problem of position estimation using accelerometer data and the estimated orientation.

3.1 Orientation Estimate

Note that the on-board gyroscopes in a strap-down IMU measure the *body* angular velocity in an inertial frame. Let us first consider the case where the vehicle operates in an inertial frame, and then make small adjustments if the vehicle operates on Earth, which is not a truly inertial environment (since the Earth is spinning about its axis). These gyroscopic measurements can be used to update the moving vehicle's orientation estimate as follows.

Assume that the IMU's body-fixed reference frame starts off at time $t = 0$ with initial orientation R^{0}_{WB} relative to the fixed observing frame (which might be an inertial frame in space, or an observing frame on earth, which is not an an inertial frame). It is often convenient to choose $R^0_{WB} = I$. Let us further assume that at time $t_1 = \Delta t$, we sample the IMU gyros, and obtain the 3 angular velocities

$$\vec{\Omega}^1 \;=\; \begin{bmatrix} \Omega^1_x \\ \Omega^1_y \\ \Omega^1_z \end{bmatrix}.$$

Assume that Δt is very small, so that we approximately assume that the axis of rotation $\vec{\omega}^1 \;=\; ||\vec{\Omega}^1||^{-1}\vec{\Omega}^1$ is fixed over the very small time interval Δt, and that the body rotates about that fixed axis by a constant rate of rotation $||\vec{\Omega}^1||$. In that case, the IMU reference frame will be oriented as

$$R^s_{WB}(t_1) = R^0_{WB} e^{\Delta\phi(t_1)\;\hat{\omega}(t_1)} = R^0_{WB} e^{\hat{\Omega}(t_1)\;\Delta t} \tag{4}$$

where $\Delta\phi(t_1) = ||\vec{\Omega}(t_1)||\Delta t$. Similarly, at $t_2 = t_1 + \Delta t$, the body will be oriented at

$$R^s_{WB}(t_2) \;=\; R^0_{WB}\; e^{\Delta\phi(t_1)\hat{\omega}(t_1)}\; e^{\Delta\phi(t_2)\vec{\omega}(t_2)} \;=\; R^0_{WB}\; e^{\hat{\Omega}(t_1)\Delta t}\; e^{\hat{\Omega}(t_2)\Delta t}$$

If we take the the limit as $\Delta t \to 0$, then it can be shown that

$$R^s_{WB}(t) \;=\; R^0_{WB}\;\exp\left(\int_0^t \;\Omega(\tau)\; d\tau\right). \tag{5}$$

3.1.1 Inertial Navigation in practice

In practice, one doesn't have access to $\vec{\Omega}(t)$ as a continuous function, as would be needed to exactly evaluation Equation (5). Instead, one samples $\vec{\Omega}(t)$ at uniform intervals, whose length is $\Delta t << 1$. If we know the vehicle's orientation at time t_k, then it's orientation at time $t_{k+1} = t_k + \Delta t$ is given to good approximation by the matrix exponential, following Equation (4)

$$\begin{aligned} R_{WB}(t_{k+1}) &= R_{WB}(t_k)e^{\Delta\phi(t_{k+1})\hat{\omega}(t_{k+1})} \qquad (6) \\ &= R_{WB}(t_k)\left[I + \sin(\Delta\phi(t_{k+1}))\hat{\omega}(t_{k+1}) + (1-\cos(\Delta\phi(t_{k+1})))\hat{\omega}^2(t_{k+1})\right] \end{aligned}$$

where

$$\Delta\phi(t_{k+1}) \;=\; ||\vec{\Omega}(t_{k+1}||\Delta t \tag{7}$$

$$\hat{\omega}(t_{k+1}) \;=\; ||\vec{\Omega}(t_{k+1})||^{-1}\;\vec{\Omega}(t_{k+1}) \tag{8}$$

$$\vec{\Omega}(t+\Delta t) \;=\; \text{measured body angular velocity vector at time } t+\Delta t. \tag{9}$$

That is, we assume that the instantaneous axis of rotation is fixed during an interval Δt, and that the rate of rotation is also constant during this interval.

3.2 Position Estimate

An accelerometers measures vehicle acceleration in body-centered coordinates, $\vec{a}^b$. To estimate position from the accelerometer data, we first transform this data to the inertial world coordinate system:

$$\vec{a}^s(t) \;=\; R_{WB}(t)\;\vec{a}^b(t)\;.$$

To obtain an estimate of the vehicle's velocity, $\vec{v}^s_{WB}(t)$, at time t

$$\vec{v}^s_{WB} = \vec{v}^s_{WB}(t=0)\;+\;\int_0^t \;\vec{a}^s(\tau)\;d\tau\;.$$

while an estimate of the vehicle's position, $\vec{p}^s$, (in the fixed inertial coordinate system) is given by

$$\vec{p}_{WB}\;=\;\vec{p}_{WB}(t=0)\;+\;\int_0^t \;\vec{v}^s_{WB}(\tau)\;d\tau\;.$$

3.3 Position Estimation in Practice

Assuming that the accelerometers are sampled at uniform intervals, spaced by time Δt, the vehicle's spatial velocity and position can be iteratively updated as:

$$\vec{v}^s_{WB}(t_{k+1}) \;=\; \vec{v}^s_{WB}(t_k)\;+\;(R_{WB}(t_{k+1})\vec{a}^b(t_{k+1})\;\Delta t \tag{10}$$

$$\vec{p}_{WB}(t_{k+1}) \;=\; \vec{p}_{WB}(t_k)\;+\;\vec{v}^s_{WB}(t_{k+1})\;\Delta t \tag{11}$$

4 Terrestrial Navigation

A reference observing frame W, which is fixed in the Earth is not truly an inertial reference frame, since the Earth is moving. Moreover, we must account for gravity on Earth as well.

Adjusting the gyro measurements: To account for the Earth's spin, we must subtract the rotation of the Earth from the gyroscope readings (which measure rotations relative to a fixed inertial frame:

$$\Omega^b_{WB}(t) = \Omega^b_{IB} - \omega^b_{IE}$$

where Ω^b_{IB} is the measured gyroscopic velocities (which measure the rate of rotation of the moving body, $\mathcal{B}$, relative to inertial space) and ω^b_{IE} is the Earth's rotation, represented in the gyroscopic frame. The corrected velocity, Ω^b_{WB} should be used in the formulas above, whereever the gyroscopic measurements were used.

Note that this effect is small, and may be of a magnitude which is below the sensitity of cheap gyros. The Earth rotates one revolution (or 2π radians) per day, which has a length of 24 *hours* $\times$ 3600 *secs/hour* $=$ 86,400 *secs*. Hence, the Earth's rotational rate is: $||\vec{\omega}_{IE}|| = 2\pi/86,400\;sec = 7.27\times10^{-5}\;rad/sec = 0.00417\;degrees/sec.$

Adjusting the accelerometer measurements:

- The accelerometer signal must be adjusted for the effect of gravity.

$$\vec{a}^s_{WB}(t) = R_{WB}(t)\vec{a}^b_{WB}(t) - g^s(t)$$

  where $g(t)$ is the gravitational acceleration (in the fixed, world, observing grame).

- For more accurate inertial navigation (when the accuracy of the accelerometer is very good), the spatial acceleration can be ajusted by the Coriolis force: $-2\omega^s_{IE} \times \vec{v}^s_{WB}$, where ω_{IE} is the Earth's rotational rate, and $\vec{v}^s_{WB}$ is the estimated velocity of the vehicle relative to a fixed observing terrestrial frame.

- The accelerometer signal can also be adjusted for centripetal acceleration if it has a very high accuracy: $-\omega^s_{IE} \times \omega^s_{IE} \times \vec{p}_{EB}$ where ω^s_{IE} is the Earth's rotational velocity, and $\vec{p}_{EB}$ is a vector pointing from the center of the Earth to the origin of the vehicle's body fixed reference frame.

5 Issues to consider

Clearly, the robot's estimate of its position will *drift* over time, as there multiple sources of error which can corrupt the position estimation process:

1. Noise in the gyro and/or accelerometers measurements.
2. "Drift" in the gyroscope (this is error in the inertial measurements due to phyisical processes in the gyro mechanism)
3. Errors in the numerical integration process.
4. Errors in the multiplication of rotation matrices. I.e., due to numerical round-off in the multiplcation process, the product of rotation matrices may no longer be a rotation matrix.
5. coupling of errors in the orientation estimate to the position estimate.